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Mathematics and the USSR: Organising a Discipline

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Declaration

I declare that the work contained within has been composed by me and is entirely my own work. No part of this thesis has been submitted for any other degree or professional qualification

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Abstract

This thesis aspires to establish a new research direction in STS. In the first chapter a literature review is conducted and the research questions are being formulated. The second chapter is devoted to presenting research findings from the archaeological, biological and brain sciences in a unified form. The various stone tool technologies are analysed, and a brief introduction follows into human evolution and the effects that artefacts had on it; then recent neurobiological research on the deeper relationships between consciousness, artefacts and the brain is presented. In the third chapter, after an introduction in the deeper neurological relationships between language and gestures, a gestural analysis of mathematical speech follows, based on visual data generated from an interview with a working mathematician; the last section examines recent research on gesture and mathematics as special cases of Roman Ingarden's aesthetic theory. In the fourth chapter, four approaches to the social history of mathematics in the USSR are presented, based on data generated from interviews with former professional Soviet mathematicians. Following a Maussian approach, the Soviet mathematical community is presented as a gift economy of scientific articles. Then, in line with a Marxian approach, the Soviet university mathematical school is presented as a factory with its own mode of self-production. In the following section, based on a Parsonian systemic approach, the Soviet mathematical community is presented as a banking system, with the scientific journals as the banking institutions. In the next section of the fourth chapter, following a Weberian approach, the mathematical community in the USSR is presented as a social estate, as separate and distinct from other Soviet social estates. The final section integrates the previous approaches and presents the Soviet mathematics

research community as a modern version of an ancient city-state. In the fifth chapter Hilbert spaces are briefly presented, as an example of the fictional universe of modern mathematics, along with some conjectured differences between Soviet and Western mathematics research. In the final chapter, the conclusions of this research project are summarised, and this thesis is presented as an instance of a proposed revised version of David Bloor's Strong Programme.

Dedication

To Olga for her patience and love.

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Chapter 1

Opening the Stage

1.1 The Bloorean Charter of New Sociology

David Bloor is, most probably, known as a major proponent of the new, at the time, wave in the sociology of scientific knowledge (SSK), after Robert Merton's original research programme in the sociology of science. SSK came to be interested more in the interactions between human cognition, scientific practice, and societal influences. More accurately, SSK researchers started developing a more ethnographic approach to scientific cognition, leaving aside theoretical approaches of a philosophically defined type of knowledge. Bloor aspired "for a conception of the natural world as morally empty and neutral" [54, p.9]. In the same way, in other words, that a primatologist would study nonhuman primates without focussing on moral issues with respect to nonhuman primate social groupings, Bloor proposed a primatology of human scientists. Instead, though, of simply observing a human primate social group of scientists, a sociologist, that is, a behavioral biologist of human primates, would approach his or her subjects as an ethnographer, that is, as a human primate studying his or her conspecifics, or rather as an intra-species participant observer.

Bloor called for a revision of Merton's social-research imperatives, and called his revised research-imperatives the *Strong Programme* in SSK. Since the

sociologists of scientific knowledge were now primatologists, or rather behavioral biologists, they had to resort to natural causality explanatory claims, as physicists, chemists, and biologists would do, adding to these the causality claims of their own field, that is, social-cultural explanations: this was the *causality research imperative*. Moreover, the Strong Programme should adhere to an *impartiality research imperative* “with respect to truth and falsity, rationality or irrationality, success or failure” [p. 7]. True knowledge, rationality, or success were, in this view, social constructions, and it would, therefore, be pointless to treat them in a different fashion from false knowledge, irrationality, or failure, or any other social construction. The *symmetry research imperative* would dictate the same causality narratives both for successful and unsuccessful types of scientific knowledge, that is, the social researcher could not invoke specific sociological explanations for specific fields of scientific knowledge; common underlying explanations, rather, both for successful and unsuccessful claims to knowledge should be proposed as explanation narratives. The *reflexivity research imperative* would make visible the fact that the intra-species participant observer would not escape his or her own socially constructed scientific claims.

There was, however, another research imperative, or rather a crypto-imperative, which seems to have eluded Bloor, and, in general, is being ignored in the relevant literature. In Bloor’s opinion, SSK had met with strong resistance within the scientific community because

[s]cience is sacred, so it must be kept apart. It is . . . “reified” or “mystified”. This protects it from pollution which would destroy its efficacy, authority and strength as a source of knowledge [p. 49–50].

Scientific knowledge, in other words, is part of the *social imaginary*, that is, the unconscious mirror image of society itself, and SSK was apparently set to *demystify* and *profane* it extensively. David Bloor’s fifth research imperative of the Strong Programme was, in fact, a call for a (continuous and repeated) secularization of science, or rather, for a *linguistification of the sacred*:

The aura of rapture and terror that emanates from the sacred, the *spellbinding* power of the [scientific] holy, is sublimated into the [social] *binding/bonding* force of criticizable [scientific] validity claims and at the same time turned into an everyday occurrence [198, p. 77, italics in the original].

Scientific knowledge, as the unconscious mirror image of the (scientific) society itself,

provides a channel through which our simplified social ideologies make contact with our theories of knowledge. It is these ideologies rather than the totality of our real social experience which might be expected to control and structure our theories of knowledge [54, p. 53].

The Strong Programme, as a research programme of linguistification of the scientific sacred, would assist sociologists in separating and isolating the ideological part from the scientific part of the various theories of knowledge dominant at the time. Ideology, though, has proved quite resistant to empirical analysis, and hard to penetrate scientifically, and separate it from the surrounding reality. Ideology

in its basic dimension it is a fantasy-construction which serves as a support for our ‘reality’ itself: an ‘illusion’ which structures our effective, real social relations and thereby masks some insupportable, real, impossible kernel [477, p. 45].

What would that “insupportable, real kernel” be that ideology would mask? In the case of science, there seems to be a possible candidate for that impossibility: the proverbial *gap in the literature*. Every scientific publication asserts, sometimes implicitly, but more often explicitly, to be “filling a gap in the literature.” But scientists have been filling this proverbial gap for, at least, over two centuries; this gap is still being filled without any pause whatsoever, and, quite probably, it will continue to be filled for quite some time in

the distant future. This gap in the literature “awaiting to be filled” seems, in fact, to be the hard core of the official, or unofficial, scientific meta-narrative, that is, the shared narrative embracing all the local narratives that each scientific field generates, and the grand narrative of all science that mobilises the scientific community into scientific social action.

David Bloor decided to break down and space out all the components of the scientific sacred by laying afresh the claim of “filling the gap in the literature”, generating, thus, a “new gap to be filled,” espousing, in this way, the scientific meta-narrative itself of his time. In other words, Bloor proposed an alternative ideological surrogate; by attempting to linguistify the scientific sacred, he created a new sacred, a new *necessary illusion*, a new *false consciousness*. Ideology, in other words, is not simply false consciousness, as the popular view has it;¹ [human] consciousness, rather, is *false anyway*, and scientific consciousness is no exception. Bloor, in other words, swapped one ideological component for another. From the moment that consciousness is false anyway, the concept of fantasy jumps into the scene in a rather unexpected way, as the white knight championing a new scientific era. If, in other words, we identify ideology with fantasy, and if we assume that what is left to scientists is the choice of one fantasy over another, then another view of scientific activity emerges out of the smokescreen of “modern post-modernist” interpretations of culture. The defining characteristic of scientific activity, that is, the [once again] proverbial scientific consensus, becomes the convergence principle of divergent social-phantasmatic apparatuses.

1.2 Reinventing the Socially Constructed

David Bloor, in fact, was a contributor to a more general trend in the social sciences, to what, actually, came to be generically called the *social construction of reality*. The term was first popularised in the social sciences by Berger & Luckman [45],² almost a decade before Bloor’s first major publication, al-

¹See for example [134, p. 1–32].

²For a history of social constructionism in the social sciences and the humanities see [296]; see also [199].

though Bloor himself never used the term. The main ideas behind the social construction of reality are basically two:

- (a) the central assumption that people make sense of experience by constructing a model of the social world and how it works and
- (b) the emphasis on language as the most important system through which reality is constructed [277, p. 892].

Social groups, in other words, continuously perform (mainly linguistic) “attempts to construct, maintain, repair, and transform [social] reality” [80, p. 24], and these attempts are being influenced by culture and social institutions.

Two more trends of social constructionism have also been present in social research. One is *discursive constructionism*, influenced mainly by the work of Michel Foucault:

For Foucault, it is *discourse* that shapes the social world . . . *Language* provides us with the tools to express meaning and therefore shapes *how* we may do so, whereas *discourse*, at least as Foucault uses the term, relates to the regulation of the *content* of *what* we say. [141, p. 11, emphasis in the original].

The other trend, which Bloor espoused, is *knowledge constructionism*:

[K]nowledge for the sociologist is whatever people take to be knowledge. It consists of those beliefs which people confidently hold to and live by. In particular the sociologist will be concerned with beliefs which are taken for granted or institutionalised, or invested with authority by groups of people . . . [We reserve though] the word ‘knowledge’ for what is collectively endorsed, leaving the individual and idiosyncratic to count as mere belief [54, p. 5].

Bloor’s social constructionism, in other words, consists in renaming collective belief, or collectively believed facts, as knowledge, and the sociology of scien-

tific knowledge is, actually, the sociological study of the collective scientific beliefs (and practices) of the various communities of science.

A major question to be raised by a natural scientist, or rather a behavioral biologist, would refer to the biological possibility of social constructions. How is it biologically possible, in other words, to socially construct realities in human social groups, that can be so varied and diverse, as many anthropological studies have repeatedly demonstrated? The second chapter of this thesis attempts to answer exactly that in two successive steps. In the first place, the major difference between the *Homo* biological genus and the rest of the animal kingdom is the *systematic use of artefacts*. The only difference, in other words, between humans, Neandertals, *Homo erectus*, and so on, and the rest of the mammals, as well as the non mammals, is the construction of artefacts. Artefacts, in fact, were a major cognitive leap in biological evolution, and this leap was the result of the interaction between the ecological environment of the long past of 2–3 million years ago, and the bodily anatomy of the *Homo* species. This evolutionary leap has been reflected in the differences between the neural substratums of the modern human brain and the brain of modern nonhuman primates, in spite of the fact that all the modern primates share a common evolutionary ancestor, and, therefore, they should share a brain of the same evolutionary origins.

The second chapter, in addition, attempts to answer another more interesting question: how the *Homo sapiens* species differs from the other *Homo* species, since the proliferation of *Homo sapiens* material culture has been unprecedented. In order to answer that question, the author has reversed the common definitions of a “modern society.” The usual definitions of modernity in the social sciences start from today’s society, and then go on to define societies that precede or, especially those that, come after the modern ones: societies of “postmodernity”, societies of “late modernity”, societies of “fluid modernity”, societies of “multiple modernity”, and so on. A lay person would be most probably led to think of two questions. After two or five hundred years, are the sociologists of the future going to start talking about “post-post modernity”, or “post-late modernity”, or “late-late modernity”, or maybe “late-post modernity” and so on? This question demonstrates the

cumbersome character of attempting to identify modern society in terms of its temporary character in the long historical time. The second question, which in fact is an observation, has to do with sociologists themselves, or at least a group of social theorists: they seem to rush to define the historical period of their time, like a teenager who is impatient in acquiring general recognition: definition of a certain historical era comes about usually by the generations of the future historians, one or two hundred years later. Sociologists of modernity seem to have confused current social trends with historical ones: currency of events is only transient, while their historicity spans at least half a century.

In this thesis, modernity, following the current biological research threads, is defined as the *historical* period that *Homo sapiens* appeared, a period marked with the concomitant appearance of cave art, dance, music, mythology, modern language, and so on. In this period, the main social activity moved from artefact construction to the emergence of the *social imaginary*, rendering the artefacts simply the *material traces* of the social imaginary. The social imaginary is the transcendental reality that made social constructions possible, and as a result sociology that does not deal with the social imaginary, is simply ethology, that is, behavioral biology. It is attempted, in other words, in this thesis, to have the research objectives of the social sciences redefined, and then continue with proceeding to a more specialised study of a particular socially constructed activity, that of mathematical research. The social imaginary rendered the modern human societies to become social (animal) groupings of gigantic proportions, both in population and geographical size. In spite, though, of the unprecedented high rate in modern technological innovation, societies still have remained rather backward, according to today's "Western" standards. The social imaginary does not change that fast, as technological innovation seems to delude us into believing, and in the end of the day, the so-called, or rather self-styled, modern/advanced societies are only about a sixth of the world's population.

The study of the social imaginary calls forth a new entity, or rather a new social agent: that of *imagination*, and its relation to the artefact. Following Bernard Stiegler, any stone tool is in fact a *prosthesis*:

By pros-thesis [sic], we understand (1) set in front, or spatialization (de-severance [sic]); (2) set in advance, already there (past) and anticipation (foresight), that is, temporalization . . . The prosthesis is not a mere extension of the human body; it is the constitution of this body *qua* “human” (the quotation marks belong to the constitution). It is not a “means” for the human but its end, and we know the essential equivocality of this expression: “the end of the human [i.e. *Homo sapiens*]” [is the artefact] [421, p. 152–153].

The stone tool, in other words, marks the appearance of time, that is, a *before*, and an *after* the construction of the tool, as well as a *before*, and an *after* its use. A tool marks the *Homo* cognitive abilities of *hindsight*, that is, remembering how a tool has been constructed in the past while it is absent at present, and that of *foresight*, that is, the end result of construction, while it is still absent from the present. Systematic construction of a stone tool, in other words, increases to a great degree the cognitive load of the brain. The stone tool, more specifically, acquires *imaginary locomotion*, and the movement of hands towards constructing it is simply the visible aspect of it.

Imaginary locomotion is, in fact, another term for *Homo* imagination as an *act of Homo-species consciousness*. Following Jean-Paul Sartre’s phenomenology of the imaginary, artefact-imagination has four basic characteristics. The first is that “the [artefact-]image is a consciousness [itself]”:

We thought [...] that the image was *in* consciousness and that the object of the image was *in* the image. We depicted consciousness as a place peopled with small imitations and these imitations were the [artefact-]images. Without any doubt, the origin of this illusion must be sought in our habit of thinking in space and in terms of space. I will call it: *the illusion of immanence* [401, p. 5, italics in the original].

The mental image of the past artefact to be a construction model, as well as the mental image of the future artefact to be constructed, *are not in our*

heads, as the common wisdom has it, but they constitute a new transcendental vision, as our *third eye*, which is itself the imaginary reality being sustained from the recent past while being projected onto the imminent future.

The second characteristic of artefact-imagination is the appearance of the “phenomenon of quasi-observation”:

Our attitude in relation to the object of the [artefact-]image could be called “*quasi-observation*”. We are, indeed, placed in the attitude of observation, but it is an observation that does not teach anything [401, p. 10, italics in the original].

When observing a stone tool in front of me, I see it only partially, and slowly I acquire a view of all its sides; during each moment of observation, though, I always have only a partial view of it. When I am constructing the stone tool, on the other hand, I am *observing* the (imaginary) stone tools of the recently visited past, as well as the intended (imaginary) tool of the soon-to-be-visited future, comparing them with the (material) stone tool under construction, and accordingly I continue with knapping the stone core: observation of the past tool, as well as of the future tool, are in fact, quasi-observations, not real ones. Although quasi-observation is, actually, observation of imaginary objects, there is still the impression that the real material stone core under hand processing is the same as the imaginary stone tools both of the past and of the future used as models of the end-product.

The third characteristic is that “the [artefact-]imaging consciousness posits its object as nothingness”:

in vain we seek by our *conduct* towards the object to give rise to the belief that it really exists; we can ignore for a second, but cannot destroy the immediate consciousness of its nothingness [401, p. 14, italics in the original].

Imagination simply points to *material nothingness*: our imaginary objects, which are right in front of us, and we can use phantom, or rather imagined,

body and hands to touch them, are simply nowhere to be found, never to be discovered; our imagined stone tools of the past and the future, absent from the present, have gone beyond space and time, yet they are so closely visible to us. Still, despite their nothingness, imaginary artefacts display the fourth characteristic of “spontaneity”:

A perceptual consciousness appears to itself as passive. On the other hand, an [artefact-]imaging consciousness gives itself to itself as ... a *spontaneity* that produces and conserves the object as imaged ... The consciousness appears to itself as creative, but without positing as object this creative character [401, p. 14, my italics].

Since stone tools spanned at least two million years, and most probably tool construction from materials less resistant to time must have preceded it, it is almost certain that the construction skills were picked up, for example, by observing³ older members in a *Homo* band society, or by having the technique shown by a more competent group member: modern humans, and therefore their evolutionary *Homo* ancestors are born with the ability to construct stone tools, in general, but social processes sustain and preserve the specificity of each stone tool. Members of many spider species, on the other hand, are born with the ability to create cobwebs, and there is no social process regulating the specificity of each web: every species creates its own distinct webs, and each individual of a species creates the same cobweb pattern as any other individual of the same species. In fact, Marx himself mentioned this human ability in his attempt to differentiate human labour from animal behaviour:

A spider conducts operations which resemble those of the weaver, and a bee would put many a human architect to shame by the construction of its honeycomb cells. But what distinguishes the worst architect from the best of bees is that the architect builds the cell in his mind before he constructs it in wax. At the end

³That is social-observational learning.

of every labour process, a result emerges which had already been conceived by the worker at the beginning, hence already existed ideally [310, p. 284].

Stone tool construction, in other words, was already social among *Homo* members of the same group, and the materiality of the tool, that is, its publicly available visibility, was sparking the spontaneity of the social imaginary of the group.

While perception, in other words, is passive, imagination is active, “spontaneous”, and *creates* its own objects out of a *material nothingness*. It could be very well argued that a stone tool is a precursor to symbolism, that is, the semiotic ability of the *Homo* genus: instead of using a stone tool just for cutting flesh, in the hands of an individual of a *Homo* species the selfsame tool can be used as a territorial mark, or even as a talisman protecting its owner from evil spirits.⁴ Whereas the (materially) symbolic function seems, indeed, to be peculiar to the *Homo* species, without the imaginary there can be no symbolism whatsoever:

the imaginary has to use the symbolic not only to ‘express’ itself (this is self-evident), but to ‘exist’, to pass from the virtual to anything more than this. [...] But, conversely, symbolism too presupposes an imaginary capacity. For it presupposes the capacity to see in a thing what it is not, to see it other than it is. [84, p. 127].

In this thesis, in other words, there is limited reference to the social construction of reality; one of its main underlying themes, instead, is the *social invention of (mathematical) reality*. Construction of reality, as a phrase, would invoke an imagery of a construction site, in which the activities of the participants, that is, engineers, builders, architects, and so on, follow a more general preset plan. The invention of reality, on the contrary, promotes a view of science in which there is no general plan to be followed by the participants, that

⁴Talismans, in particular, are considered to have been constructed only by *Homo sapiens*.

is, the mathematicians; the mathematicians, rather, take research initiatives, and the whole scientific edifice is the result of each individual imagination, while mathematical proof is the practice-medium of coordinating the imaginative socio-technical interventions into a common social reality. While the social construction approach presents scientists as following a general preset plan of construction, the social invention approach, proposed in this thesis, presents social reality as the result of scientists' research initiatives. This approach, in other words, promotes a view of mathematics not as a science, but as a community of artists whose social reality is coordinated by mathematical proof.

1.3 Scientific Knowledge and its Discontents

Barry Barnes was working alongside Bloor and developing SSK from another viewpoint. Drawing on Habermas (see [197]), Barnes claimed that the interests of scientists dictate the directions of scientific knowledge, even though scientists themselves might deny such an influence:

the 'disinterested evaluation' of knowledge is in most contexts a harmless enough formulation, which can be taken as practically equivalent to 'evaluation in terms of an authentic interest in prediction and control'. [32, p. 91].

Donald MacKenzie picked up on this viewpoint, as an explanatory narrative in a scientific dispute between two statisticians, Pearson and Yule, that arose at the turn of the nineteenth century in Britain:

these different cognitive interests arose from the different problem situations of a statistician whose primary commitment was to a research programme in eugenics and a statistician who lacked any such strong specific commitment; and finally, that eugenics itself embodied the social interests of a specific sector of British society, and not those of other sectors. Thus differing social interests can

be seen as entering indirectly, through the ‘mediation’ of eugenics, into the development of statistical theory in Britain [301, p. 71].

Cognitive interests can, indeed, be considered social constructions;⁵ MacKenzie though, in his narrative, continued to view mathematical statistics, rather than mathematics itself, as a transcendental and absolute reality, independent of the mathematicians themselves: this thesis will attempt to bring this view into revision.

Harry Collins, another early proponent of SSK, rejected Bloor’s *causality* and *reflexivity* research imperatives, but kept the *symmetry* and *impartiality* imperatives (see [100, p. 216]). Collins, on the other hand, proposed the *Empirical Programme of Relativism*, which would consist of *three stages*:

- (1) revealing the inevitable openness or interpretative flexibility of scientific results;
- (2) examining the social processes that are employed to close debates over results;
- (3) investigating the connection between these processes and social forces beyond the immediate community of scientists [472, p. 29].

Collins’s research agenda could have, very well, included one of the most debated issues in the history of modern mathematics: set theory, and the introduction of *infinity*. As we will briefly see in chapter 5, transfinite numbers met with fierce opposition, and their acceptance became wider after Hilbert’s official endorsement: the acceptance or not of the infinity became a major controversy at the end of the nineteenth century, because both the introduction of infinity, as well as its abandonment were acceptable as mathematical realities; mathematical reality could be interpreted, in other words, both ways. Later on, infinity led to standard methodologies in mathematics, and today there is no widespread dispute over it in the mathematical community. Acceptance, rather, of mathematical infinity has simplified many

⁵Habermas went even deeper: interests, in his theory, were unconscious, and (Freudean) psychoanalysis had to be employed to “liberate” the individual from these.

methodologies. There have been some efforts to develop mathematics by dropping the concept of infinity, such as nonstandard analysis, but these have remained rather marginal curiosities, than widespread practices. Although Collins himself never conducted any sociological study on mathematics, his empirical programme of relativism could indeed prove useful in studying, for example, the controversy over the acceptance or rejection of transfinite numbers in mathematics at the turn of the nineteenth century, and how this was solved by later international conferences and research practices. Still, though, he never considered the social imaginary as a social agent, and never tried to document the relationship between materiality and imagination.

Bruno Latour and Steve Woolgar, in one of their early sociological studies of science, attempted to study the “construction of scientific facts” in a scientific laboratory:

From their initial inception members of the laboratory are unable to determine whether statements are true or false, objective or subjective, highly likely or quite probable . . . Once the statement begins to stabilise, however, an important change takes place. *The statement becomes a split entity.* On the one hand, it is a set of [sound or written] words which represents a statement about an [imaginary] object. On the other hand, it corresponds to an [imaginary] object in itself which takes on a life of its own. It is as if the original statement had projected a virtual image of itself which exists outside the statement [268, p.176, italics in the original].

We can see in Latour’s and Woolgar’s account that the scientific statement acquires temporary imaginary locomotion in a rather subtle and imperceptible way: the imaginary object detaches itself from the (material scientific) statement, and starts living “a life of its own”. This is typical of Latour’s style, a style of *sociological magical realism*:

One of the unique features of magical realism is its reliance upon the reader to follow the example of the narrator in accepting both

realistic and magical perspectives of reality on the same level. It relies upon the full acceptance of the veracity of the fiction during the reading experience, no matter how different this perspective may be to the reader's nonreading opinions and judgements [61, p. 3–4].

Latour's magical realism was his way of describing the imaginary locomotion of artefacts; in this thesis, the same will be demonstrated by capturing, though, the hand gestures of a mathematician during his mathematical descriptions.

One of Bloor's main aims in his strong programme was to establish the possibility of an alternative mathematics: in his first major publication he attempted to establish that there can be an alternative mathematics, and he tried to demonstrate that ancient Greek mathematics, especially the version of the Pythagorean secret society of philosophers, were indeed an alternative one (see [54, p. 118–125]). While Bloor's earnestness cannot be doubted, he did miss some important points. The first was that we are not in a position any more to judge as scientists as to how alternative was the mathematics of ancient Greece: only a genuine ethnographic research could reveal the social imaginary underlying the material reading of numbers, whether rational, or irrational such as $\sqrt{2}$; modern mathematics has been irreversibly socially contaminated by the ancient Greek version of it. Another more important point he could have emphasised more was the inventiveness of the mathematicians themselves to reverse the reality of mathematics and mathematical practice, an example of which is the so called “noncommutative geometry”.

During the author's first master's course in mathematics, he stumbled upon what has come to be called “noncommutative” [or sometimes “quantum”] mathematics. An algebra in mathematics is an algebraic ring with the operations of addition and multiplication, over an algebraic field. In more simple terms, an algebra is a vector space with an extra multiplication operation defined among its vectors. The algebraic ring, for example, of $n \times n$ matrices is an algebra over the fields of real or complex numbers. The algebraic ring of continuous, or measurable, (real or complex) functions is an algebra over the

(real or complex) numbers. While, though, the algebra of continuous, as well as that of measurable, functions is commutative, that is, when multiplied they *commute* ($fg = gf$), the algebra of $n \times n$ matrices is *noncommutative*, that is, the matrices do not commute ($AB \neq BA$). The most studied noncommutative algebras are the operator algebras, that is, algebras of “infinite” matrices, whose applications are mainly in quantum physics. C^* ⁶ algebras, in particular, are, usually operator, algebras with a *norm*, that is, in rough terms, a modulus function, so as to define limits on it.⁷

A special theorem, the Gelfand-Naimark theorem, states, in simple terms, that any commutative C^* algebra A is *isometrically $*$ -isomorphic* to the C^* algebra $C_0(X)$ ⁸ for some locally compact Hausdorff space X ; if X is a compact Hausdorff space, then $C_0(X)$ is simply $C(X)$, that is, the (Hilbert) space of continuous functions on X . Leaving aside the technicalities, there is an important conceptual result involved in this theorem: the “isometrically $*$ -isomorphic” means that the Hausdorff space X of the theorem is *unique*, and many important topological properties of X correspond to certain corresponding algebraic properties of A . In other words “*all* topological information about X is stored algebraically” in the algebra A [455, p.24, italics in the original]. The question, now that arises, and started to arise after the publication of the Gelfand-Naimark theorem, is what happens when we have a *noncommutative C^* algebra* to deal with, such as the algebra of the complex $n \times n$ matrices? The Gelfand-Naimark theorem refers *only* to commutative ones. The answer given by the community of mathematicians is that a noncommutative C^* algebra should “logically” correspond to a “noncommutative locally compact Hausdorff space”, whose set-theoretic, that is point-, representation we simply do not know. In simple terms, noncommutative topological spaces are “point-less”, that is, we cannot draw *any* graphs of functions on them. The corresponding noncommutative measure spaces, for example, are the *von Neumann algebras*,⁹ and there is a whole

⁶Pronounced C-star.

⁷Denoted by $\|A\|$, and pronounced “norm of A”.

⁸That is, the space of complex functions *vanishing at infinity*, and defined on X .

⁹See [432, Vol. II].

field of noncommutative probability theory and stochastic integration,¹⁰ as well as, noncommutative ergodic theory, dynamical systems, and differential (Riemannian) geometry.¹¹

Noncommutative mathematics is indeed an alternative, and quite revolutionary one could say, mathematics that rendered, in fact, the foundational set theory, redundant, and promoted a new calculus with different and more refined types of infinitesimals: the usual (Newton-Leibniz) infinitesimals are the compact operators in a infinite-dimensional separable Hilbert space, plus additional types of infinitesimals of many different orders (see [105, p. 26]). Due to the fact that the *commutator* of two elements A, B of a noncommutative algebra \mathbf{M} is a nonzero element of \mathbf{M} , that is $[A, B] = AB - BA \neq O_{\mathbf{M}}$, noncommutative mathematics is referred to, sometimes, as *quantum* mathematics, because the nonzero commutator is the basis of the Heisenberg's uncertainty principle in quantum physics: the C^* algebra of $n \times n$ matrices is, in other words, a quantum topological space.

The example of noncommutative mathematics demonstrates, in the author's opinion, that Bloor's alternative mathematics can be found within modern mathematics itself, as it reinvents itself: there is no need to speculate as to the resourcefulness of mathematicians themselves. Mathematical creativity, in general has been ignored by the sociology of mathematics, that is, mathematics as an artistic genre. Donald MacKenzie, for example, has conducted a comparative study of proof in mathematics and in computer programming [303]. Although MacKenzie provides a convincing account of "informal proofs" among mathematicians as compared to "formal" and automated proofs conducted by computer programs, he still missed, in the author's opinion, the creative process of mathematical creativity in the invention and formulation of a proof. Even a computer program that computationally proves a certain class of mathematical theorems has to be invented by the programmer himself or herself.

Reviel Netz, on the other hand, [340] has focussed on the particular materiality of mathematical proof in ancient Greek mathematics: the *lettered*

¹⁰See [324, 352].

¹¹See [432, Vol. III], and [105].

diagram. The lettered diagrams, according to Netz,

– in the specific way they are used in Greek mathematics – are the Greek mathematical way of tapping human visual cognitive resources. [Moreover] Greek mathematical language is a way of tapping human linguistic cognitive resources. These tools are then combined in specific ways. The tools, and their modes of combination, are the cognitive method. [340, p. 6].

While human mathematicians, in other words, use the materiality of their surrounding, and possibly artificially modified, ecosystem, such as lettered diagrams, computers, on the contrary, conduct automated proofs with the use of materiality provided to them by the programmers, that is, electrical signals. In other words, both the mathematical proofs, as well as the automated proofs have as their agents the humans.¹² The human agency of mathematical, as well as, automated proofs, is possible only because of the mathematicians' imagination: there is, in other words, a deeper relationship between mathematical proof, and the subject of proof and the material resources for proving which have been left out of sociological empirical research.

The view of material resources employed in proving a theorem was taken up by Eric Livingstone who approached mathematical proof from ethnomethodology's point of view [293]. Livingston's approach

formulates and, in a certain sense, solves the problem of the foundations of mathematics as a problem in the local production of social order [293, p. x].

Livingston takes up Gödel's theorem and attempts to connect the written formulation of a mathematical proof to the imaginary reality of the proof by speaking

of the intrinsically tied pair the-material-proof /the-practices-of-proving- to-which-that-proof-is-irremediably-tied [sic] [293, p. 145].

¹²At least as long as there is no artificial intelligence yet.

In this way he refers to a “material proof” as “as the achievement of a prover’s lived-work”, in a similar way that a written musical score is a musician’s live performance. This view is taken up, as well, in this thesis, by focusing on what happens when the usual technical materiality employed in a proof, such as paper, pencil, and formulas, or chalk, blackboard, and diagrams, is missing; when, in other words, the more “natural” materiality of hand gestures takes up the role of “proving by explaining the proof”.

Sal Restivo has developed a very interesting viewpoint, quite similar to some aspects of this thesis. Social organisation over objectivity, according to Restivo, is concreted over by two permanent everyday processes:

Total immersion in, commitment to, and subordination to an organization, institution, or community undermines objectivity – even if the organization, institution, or community is oriented to goals of “discovery and explanation”. On the other hand, an extremely weak coupling between individual and collectivity will isolate the self and undermine the ability of the individual to communicate with others. Striking a “balance” between these extremes in the interest of objectivity leads to some form of anarchistic (mutual aid) system [382, p. 132].

The creativity of the individual scientist is, in fact, the act of “striking this particular balance” between social isolation and group membership which leads to a more anarchistic organisation of science: the scientific enterprise is basically devolved from the state, that is, from a state authority, and mainly decentralised from an internal central authority, and its operations expand as the result of the initiatives that each individual scientist takes (see also Restivo’s more recent book on scientific anarchism [383]). Pure mathematics, in Restivo’s view, is then the mathematics of a large group of scientists which has been institutionally insulated from other groups of a society, have become internally differentiated and “form a community oriented inwardly and develop a set of community ideals” [382, p. 138]. He mentions, as an example, that ancient Greek mathematics was the result of the social processes of public discourse in Athenian democracy, while the Babylonian culture, al-

though they had scribes with very good algebraic knowledge, never developed a theorem-proving culture similar to the Greek one. Mathematical proof, in other words, according to Restivo, becomes a Greek proof proper, rather than a Babylonian one, when it is being exposed to the public criticisms of an audience, and the authority of the individual prover becomes irrelevant, and therefore, invisible.

1.4 The Imaginative Lives of Scientific Others

Bettina Heintz, in her doctoral dissertation, has presented some interesting results on the sociology of mathematics. She notes, quite early in the text, that

[i]n contrast to other disciplines, which disintegrate into various and partial contradictory theories, the edifice of mathematics is still being formed as an interconnected whole. In view of the enormous specialisation, [...] this [scientific] coherence is by no means natural. Mathematics is a collective product, but not centrally coordinated. There is no authority that would take care of matching individual [scientific] results [210, p. 19, my translation].

She does note, though that there are still new connections being “discovered”, and new theories being proposed which cross over fields, the most striking example of which has been Alain Conne’s noncommutative geometry research initiative,¹³ which has involved operator algebras, algebraic geometry, differential topology, Riemannian geometry, category theory, and homological algebra, all fields once separated and largely unknown to mathematicians outside of each one of these fields. Although Heintz touched upon the idea of scientific [creative] anarchism as an organising principle of science, which Restivo has recently proposed [383], she fell short by insisting on the inflexible scientific consensus among mathematicians:

¹³Heintz does not mention this; she dealt mostly in her book with algebraic topologists.

[i]n contrast to other sciences, it seems that there is no interpretative flexibility in mathematics. The inferences of mathematics are conclusive. Anyone who holds on to the rules of mathematical methods will reach the [self]same results [210, p.20, my translation].

Heintz, in the end, concludes that “[m]odern mathematics is marked by features which hardly allow any room for sociological analysis” [210, p.274, my translation]. Although that, of course, is what many non-mathematicians might think, it is a conclusion undermined by continuous innovations in mathematics; and these innovations, in fact, do not involve mathematical proof itself, but new mathematical entities which override the inflexibility of previous proofs. The most characteristic example, in this case, is Riemannian geometry. While Euclidean, and non-Euclidean geometries were separate disciplines during most of the nineteenth century, Bernhard Riemann unified them under the two mathematical concepts of the *geodesic* on a surface, that is, the curve that connects two points on a surface with the shortest distance, and the *[Riemannian] metric*, that is, the “rule” of measuring the distance through a curve. Einstein picked up on this new geometry named after Riemann, at the beginning of the twentieth century, and formulated his general theory of relativity, a new theory of mechanics that rendered Newtonian, and, later on, Hamiltonian, mechanics special cases of relativity theory, that is, special cases in mathematical terms, not in physical terms. In terms of physics it was indeed a scientific revolution which made Einstein, overnight, into a media superstar; in terms of mathematics, though, Newtonian mechanics became an undergraduate research assignment.

Heintz ends up in the conclusion that mathematics is not amenable to sociological analysis because, in the author’s opinion (and this in general happens with the sociology of mathematics) mathematics is an “abstract”, that is a highly imaginative, discipline and the sociologists with a deeper understanding of it are very few. It has been noticed, some thirteen years ago, by MacKenzie, that “the sociology of mathematics is sparse by comparison with the sociology of the natural sciences” [303, p.2]. The situation today has not changed much. The idea of today’s mathematicians as modern shamans,

proposed in this thesis, intends to promote the difficulty of an outsider to be initiated to the shamans' spirit world, the "blindness" that a social researcher suffers from, when attempting to "go native". What is exactly Einstein's general theory of relativity? It is a special case of Riemannian, or rather semi-Riemannian, geometry, or is it a groundbreaking gravitational theory? Is noncommutative probability a generalisation of probability, or is it functional analysis "at large". Moreover, how can a social scientist answer to these questions, when he or she is not himself or herself a trained mathematician? An if he or she is, indeed, a trained mathematician, to what extent must he or she be trained: to an undergraduate level? To a postgraduate level? Or to a postdoctoral level with a few mathematical publications in his or her portfolio?

A very interesting answer to these emerging questions has been provided by Peter Galison, who, in his historical study of high energy theoretical and experimental physics, asserted that

in focusing on local coordination, not global meaning, I think one can understand the way engineers, experimenters, and theorists interact. At last I come to the connection between place, exchange, and knowledge production. But instead of looking at laboratories simply as the place where experimental information and strategies are generated, my concern is with the site –partly symbolic and partly spatial– where the local coordination between beliefs and action takes place. It is as domain I will call the *trading zone* [162, p. 782, my italics]

Galison was interested in how the experimental physicists were communicating with the theoretical ones: the former were using mostly images from experimental measurements, the latter were mostly focused on the mathematical aspects. He proposed, as a result, the metaphor of the trading zones which are very common in the trading practices among landowners and peasants in the southern Cauco valley of Colombia:

For the landowners, money is "neutral" and has a variety of natu-

ral properties; for example, it can accumulate into capital –money begets money. For the peasants, funds obtained in certain ways have intention, purpose, and moral properties. [...] So when we narrow our gaze to the peasant buying eggs in a landowner's shop we may see two people harmoniously exchanging items. [...] Out of our narrow view, however, are two vastly different symbolic and cultural systems, embedding two incompatible valuations and understandings of the objects exchanged [162, p. 804].

Galison, in other words, talked about different social imaginaries, whose point of contact was located on the common intersection of these imaginaries which is sparked by the common trading practices between peasants and landowners. The important aspect of the trading zones, for this thesis, is the materiality of the practices: the moving bodies of the exchanging agents are mutually visible during the trading, the money and products are mutually visible while being exchanged¹⁴, and, in general, the whole social activity of this exchange is mutually visible while taking place and can be mutually recalled, later on, in a conversation between the exchanging parties. Materiality, or rather moving materiality, in other words, creates a local material public sphere, which coordinates the social activity, and sparks a common local imaginary between the exchanging agents. Materiality, in other words, acts as a router in a computer network: although one computer can have installed MS Windows operating system, and another one Mac OS operating system, two operating systems incompatible with each other, when they are being connected through a Cisco router, the operating system of which is Cisco IOS and is different from both MS Windows and Mac OS, both the MS Windows and the Mac OS can perfectly communicate with each other through the router without any problem whatsoever; they both share a common communication protocol with the router. Material practices, in other words, are the (network) routers between (otherwise incompatible) social imaginaries. This human ability, that is, the sparking of a *local* social imaginary by material practices is a *Homo sapiens* ability, not shared by any other animal species. Material practices, in other words, are, in fact,

¹⁴Even in a credit/debit card transaction the card itself is visible

the *material traces* of the locally overarching social imaginary; and the most often neglected material social practice is the social practice of *human sound language*, and not simply *human language*.

In addition, Harry Collins and Robert Evans, in proposing a new direction, a “Third Wave”, in the STS discipline, have tried “to shift the focus of the epistemology-like discussion from *truth* to *expertise and experience*” [101, p. 236]. They divide expertise into three broad categories:

1. *No Expertise*: That is the degree of expertise with which the fieldworker sets out; it is insufficient to conduct a sociological analysis or do quasi-participatory fieldwork;
2. *Interactional Expertise*: This means enough expertise to interact meaningfully with participants and carry out a sociological analysis;
3. *Contributory Expertise*: This means enough expertise to contribute to the science of the field being analysed; [101, p. 254].

Although there are many research objectives shared by Collins & Evans’ research proposals and the present thesis, the main difference lies in language and its definition. While Collins and Evans implicitly accept the definition of language as the human ability to speak about, for example, science without going into further detail, this thesis espouses the current implicit consensus in archaeology that language speaking, artefact construction and, quite possibly, hand gesturing are material expressions of one unified cognitive ability that could be called, anew, *human language*.

The second chapter initiates, in fact, the reader into the relevant archaeological, biological, and cognitive research literature, in order to demonstrate that empirical research in the social imaginaries is not only lacking, but also a new research imperative has to be observed. Expertise and experience, in other words, are *embedded in the social imaginary*, and in this thesis this is demonstrated in two ways: the first, presented in the third chapter, is the conduct of gesture analysis as a material, that is, visible to the social researcher, evidence of the *phantasmatic support* of mathematical discourse; moreover, it is demonstrated, in the fourth chapter, that in the Soviet case,

mathematics was organised as a *gift economy*, a type of economic organisation that is still latent and alive even in modern science. Reformulating Collins and Evans' categorisation in this thesis's terminology, interactional expertise would mean the *ability to observe*, but *not to participate economically in*, a certain expertise group, while contributory expertise would be the *capacity to donate*, and thus *to participate economically in*, the practices of a certain scientific expertise group. Separating expertise from its imaginary dimensions is not only artificial, but also rather detrimental to scientific research, because it presents a limited view of science by cutting off its phantasmatic dimension. Let us not forget the phantasmatic dimension of money: we do not eat money, we do not drink money, we do not dress ourselves in money; yet we are made happy by money, we are ready to engage into a fight for money, and we get anxious about money. What moves societies forward (or backward), as a general rule, is not its material culture, but the phantasmatic support of that culture.

An example of exploring the phantasmatic support of a scientific theory has been offered in Loren Graham and Jean-Michel Kantor's monograph on pre-revolutionary Russian mathematics: they present a history of set theory in pre-revolutionary Russia, emphasising the social-imaginary aspect of mathematics [183]. Descriptive set theory in mathematics¹⁵ is an intellectually demanding subject: it deals with *analytic* and *coanalytic sets*, which are infinite and quite difficult to handle.¹⁶ According to Graham & Kantor, nascent research on descriptive set theory in France, a major mathematical centre in the beginning of the 20th century, was becoming something close to a disappointment. When the theory reached the scientists of imperial Russia, it was welcomed by those involved in a mystical religious movement, still alive even today, which had been, and still is, condemned as heretical by the Russian Orthodox Church: *Imiaslavie* or *Onomatodoxy*, that is, Name-Worshipping. The main idea behind Name-Worshipping is that (the Christian) God, was the same as his name when pronounced, a religious practice which, according to Graham & Kantor enhanced mathematical creativity:

¹⁵Different and more advanced from Cantor's (also called naive) original set theory.

¹⁶For a modern mathematical treatment of analytic and coanalytic sets see [235, p. 85–87]

While the French were constrained by their rationalism, the Russians were energized by their mystical faith. Just as the Russian Name Worshippers could “name God,” they could also “name infinities,” and they saw a strong analogy in the ways in which both operations were accomplished. A comparison of the predominant French and Russian attitudes toward set theory illustrates an interesting aspect of science: if science becomes too cut-and-dried, too rationalistic, this can slow down its adherents, impeding imaginative leaps. [183, p. 190].

Today descriptive set theory is used in other mathematical fields such as operator algebras and (noncommutative) ergodic theory, fields which are very demanding, even for postgraduates under supervision. Today, in other words, there are more refined and more enriched versions of infinity in modern mathematics, than those proposed by Cantor at the end of the nineteenth century. The mathematicians, though, specialising in operator algebras, or in (noncommutative) ergodic theory, as well, as in other fields, still need imagination to approach these phantasmatic scientific realities, yet the descriptive artefacts of these realities available are only mathematical formulas. The fifth chapter is devoted to a similar mathematical subject: Hilbert spaces, that is, infinite-dimensional mathematical imaginary entities, which are of fundamental importance in developing mathematical models for modern quantum physics.

In conclusion, this thesis started out to answer two questions: How *is* mathematics possible? How was *Soviet* mathematics possible? A widespread purview is that mathematics is something given and taught from a very early age in a pupil’s career, and therefore there is no question posed in the first place: “mathematics has to do with numbers”, is the usual view, inculcated in lay perception, as well as circulating in social science circles. The author, having being a postgraduate student of mathematics himself, has always been dissatisfied with this kind of lay perception, especially among practitioners in the social sciences; in the end of the day the perception that “mathematics has to do with numbers and figures” is a high school pupil’s perception, and, therefore, not appropriate for a social scientist. Moreover,

influenced by the ancient Greek geometry and its methodologies of proof, another stereotype of mathematics has become prevalent among philosophers and social scientists of science: “mathematics is the science [and maybe the craft as well] of proof”. One of the objectives, therefore, of this thesis is to promote a more balanced perception of mathematics, and demonstrate that modern mathematics has very little to do with the usual lay perception of a number, and proof is only an incidental, rather than an essential part of modern mathematics.

As to the question of Soviet mathematics, another view is prevalent among historians of science and of mathematics: “Soviet mathematics is Andrei Kolmogorov, and Andrei Kolmogorov is Soviet mathematics”, neglecting always the simple practical fact that only one person could never have proved on his own thousands and thousands of mathematical theorems of major importance. Most of the available literature on the history of Soviet mathematics is in fact a hagiography of Kolmogorov, as if after Kolmogorov’s death all of the Soviet mathematical research had come to a complete halt. This view has been adopted by most of the members of the community of the former Soviet mathematicians as well, due to a certain institutional and personal modesty on their part. Soviet mathematicians have always disregarded their own contributions in favour of Kolmogorov’s, not without a reason though: in practical terms he did set, in most of the cases on his own, the institutional foundations of mathematics research in the Soviet Union. Still, without Kolmogorov’s intellectual descendants, all of his edifice, and efforts, would have become dust in the winds of time and history. Kolmogorov’s descendants not only preserved his vision alive, but they themselves expanded it to unprecedented proportions: the author thinks that both them, and the rest of the world need to become aware of this.

1.5 Spelling Out The Questions

This particular thesis is scheduled to be the first in a series of future research, as a an imperative call to *return to Talcott Parsons*. Parsons was originally

known to the author by social rumour, rather than by personal study of his theories; until one day he stumbled upon *The American University*, one of Parsons's last books. When, in particular, the author started reading a chapter mentioning the inflation and deflation of scientific knowledge, calling the universities banking institutions, and the university professors as their fiduciaries, the author was simply and genuinely flabbergasted: how was it possible for a book that old, published in 1973, to sound so recent, in the light of the recent global economic recession of 2008–2015? How was it possible for modern sociologists to have forgotten about Parsons in general, as well as *in toto*, placing him on a cadre of classical authors in sociology, but to a status similar to that of a dinosaur: a museum exhibit of the sociological Jurassic period, interesting to know of, but extinct in today's data-oriented sociological research. A second major impetus for this thesis was another discovery in the dust aisles of the University of Edinburgh's old library books: André Leroi-Gourhan's *Gesture and Speech*. Leroi-Gourhan was a noted and highly respected French archaeologist and palaeoanthropologist with major interests in aesthetics and technology. His two-volume book *Gesture and Speech*, published in 1964–1965, was a milestone for archaeology: he proposed the idea that gestures, speech, and stone tools were three sides of the same coin: language. If Parsons today is a museum exhibit, then Leroi-Gourhan has not even made it to the footnotes of modern English-speaking sociology. The STS community, in particular, would benefit a lot from Leroi-Gourhan's theories, since he has proposed a very radical view of social constructionism.

In a Parsonian spirit, the study of Soviet mathematics would necessitate a start from the micro level, then moving to the meso level, and ending up with the macro level. The author decided to add an extra level, to study the “quantum physics proper” of social interaction: the *pico level*, that is, the level of sociology going “under the hood”, or, to use a computer-programming expression, what happens when “close to the metal”, that is, the “deep guts” of the [brain's] hardware system. The prefix *pico* was inspired by the work of the psychiatrist and behavioural economist George Ainslie. Ainslie coined the term *picoeconomics*, and his “approach is based on analysis of competing interests within the individual over time” [328, p. 116] (see also [2, 398]).

While, in other words, STS so far has been studying negotiation of knowledge within groups, i.e. in microeconomic terms *within a household*, this thesis attempts to present the view of multiple negotiations of knowledge within the individual scientist. And while Ainsley uses the term *motivational states* to denote the intrapersonal conflicts and negotiations happening over an individual economic decision, this thesis employs the term *states of consciousness* to denote the intrapersonal conflicts and negotiations happening over an individual knowledge assertion. The first, therefore, question that was set, in the beginning, was the following: what exactly *is* human language, and how is it related to materiality and technology? The second chapter starts off by answering this question. At the pico level one would expect universal phenomena happening not only across different societies, but also across archaeological time. Following Leroi-Gourhan's ideas, the second chapter starts with stone tools, not as the first sign of technology, but as the most sustainable sign of language, already present in early *Homo* species: what is the (brief) history of stone tool technology based on archaeological findings up to this day? Then, the following section presents a brief outline of the modern theory of biological evolution. The sections on stone tool technology and evolution theory are well known to any undergraduate student in archaeology, but mainly unknown to the STS community, and that is the reason for including them in the thesis. The next section, in fact, answers the subquestion: what is the neurobiology of language? In this section, we will see that, in fact, language, gestures, and stone tool use are hardwired on the same regions on the human brain: and this is, most probably, the point that Leroi-Gouran's early insights are today being validated. The last section answers another subquestion very important for studying modern research mathematics: what kind of new experience did the appearance of stone tools, and materiality, in general, have brought about? The answer given is the rise of *imagination*, and the appearance of the *sociotechnical imaginary*. The mathematical *sociotechnical* imaginary, in particular, is not visible, in its totality, to the lay eye.

Moving from the picosociology of imagination to the microsociology of mathematics, the third chapter answers the very important question for this thesis:

how can mathematical artefacts such as geometrical objects of four, five, and in general n dimensions, become visible to the specialist mathematician? Speech is not enough any more in describing certain mathematical objects, such as a Hilbert space, or an algebraic variety, or an Einstein manifold. In this chapter, therefore, again following Leroi-Gourhan's line of thought, the author decided to present gesturing while describing certain mathematical objects during an interview with a mathematician. Since the reader was already initiated in modern evolution theory, archaeological developments, and neurobiology during the reading of the second chapter, it would be much easier to present current research on gestures and speech, in one section. The following section then answers a quite intriguing question: how is gesturing, speech, and mathematical description related to one another? The author draws on material from video recordings he conducted during an interview with a mathematician, a pioneer in his field. The next section then answers another fascinating subquestion: what is, in fact, the relation between mathematics, art, and aesthetics? In this section, it is demonstrated that current research on gestures is, actually, reminiscent of the aesthetic theory of Roman Ingarden, a very prolific Polish philosopher of the first half of the twentieth century.

The fourth chapter moves on to the levels of mesosociology and macrosociology; it answers, in other words, how was the social system of Soviet mathematics, in particular, organised? A very striking finding during the author's field research was that Soviet mathematics was organised as a gift economy. The beginning section of this chapter analyses, in a Maussian spirit, the following subquestion: how a gift economy, and, in particular, the gift economy of Soviet mathematics led the individual to "latch onto" the social system of science as a reward system? Then, the following section, answers, in a Marxian way, the following subquestion: what was the mode of academic mode of self-production of Soviet mathematics? In a Parsonian spirit, then, the next section, answers another interesting subquestion: how was the banking system of science in Russia organised? And, finally, the following section, in a Weberian fashion, answers the subquestion: what made the community of Soviet mathematicians a social estate, rather than, a scientific community?

The final section, examines the very interesting question: was the social system of Soviet mathematics a Soviet version of a city-state or a postmodern nation-state? English language, in particular, is very helpful in this way. The words state and estate have a common etymological root. The social estate of Soviet mathematicians had its own system of rewards: it was a gift culture; it had its own mode of industrial self-production: the universities and the research institutes were, and still are, its factories; it had its own banking system: the scientific journals were its banks and the scientific articles were its monetary coins. The element that kept the social estate of mathematicians from becoming a state proper was Weber's definition of a state: it lacked the monopoly on violence.

The fifth chapter of this thesis examines the following question: what was, and still is, the material culture of Soviet mathematics? In this chapter, following once more Roman Ingarden's aesthetic theory, we study the levels of reading a mathematics text. Three innovations, in particular, of twentieth century mathematics are presented, as close as possible, to the way they would have been presented to a working Soviet undergraduate mathematician: countable infinity, metric functions, and Hilbert spaces. This section is intended mostly for the uninitiated, as an instance of artefacts which cannot be grasped in a conventional sense, such as these in a museum. This chapter, in fact, answers the question: how is the psychoanalytic concept of *phantasy* related to reading and understanding a mathematical proof? Another, more subtle, subquestion is: why should we consider mathematicians as a community of shamans, rather than scientists? If shamans are the only ones in a community who see the spirit world, what exactly is that spirit world of mathematicians? The deciding principle, in fact, in this chapter is *blindness*: the lay reader will be blind to the proof, that is, unable to *see* the proof. And another very interesting subquestion of this chapter: is there a universal way of proving, or does each field of mathematics have its own ways of proving? A proof in mathematical logic, in fact, is very different from a proof in functional analysis, and both of them are very different from a proof in algebraic number theory. The last section raises a very interesting issue: were there any cultural differences in understanding and proving between Soviet and

Western mathematicians?

Chapter 2

The Spectre of Artefacts

2.1 Introduction

A spectre has been haunting human society all along: *the spectre of artefacts*. It has been on the background pulling the strings of historical movement since the beginning of human time, since the dawn of consciousness. Prometheus Bound became one of its mostly known victims for his successful attempt to bring humankind down to its technical destiny. Since then its victims have increased exponentially, embracing in due arrogance the prophets, the priests, the proletarians, the revolutionaries, the citizens. Any resistance has been rendered futile; any escape became a dream; any freedom has turned into an illusion offered today laboriously and diligently to an unsuspecting public by our politicians, our patriotic and populist politicians. Nobody has ever been able to question this omnipresent merciless master, this Olympian demigod, this archangel without a soul. It surrounds with its coldness all of society's most illustrious activities: in war there are guns; in disease there are medicines; in science there are books; in religion there are shrines. Judgment day, the nuclear apocalypse, is nowadays one button away. Even human copulation, the most intimate of moments, has not totally escaped the infiltration of barrier devices. Every human activity has been contaminated by the spectre of artefacts, which lives on the twilight of perception, on the

margins of consciousness, within our *technical* human essence. Historians of technology have failed to illustrate this unseen reality, focusing on superficial arguments. Sociologists of technology in their turn renewing the effort to reinvent the wheel countless times, they have ended up themselves running in a theoretical hamster wheel, exercising their intellectual prowess without offering any research breakthrough of practical importance. Fortunately for this tortured field of science and technology studies (STS) its saviour has arrived, rather late admittedly, and it has a commonplace, and boringly bourgeois, name: *archaeology*.

Archaeology today has become a very mature science, and indeed a very sophisticated one. This fact has been hastily neglected by the field of STS, maybe out of the enthusiasm of a newly-founded field, maybe out of the arrogance of a novice. Nevertheless, the birth certificate of STS was issued by archaeology, since it has been the only science studying technology up to 2.5 million years ago. Due to the increased influence of philological studies, or the linguistic and literary studies as they are nowadays called, the general public of the social sciences and the humanities has been deeply misinformed on a crucial matter: *history started not with the appearance of written monuments but with the appearance of artefacts*; prehistory has never existed, except only as a useful taxonomy created because of division of labour. The artefact is a *symptom of consciousness*, a symptom of the emergence of time on social groups of the biological genus *Homo*. This and other later and recent scientific and theoretical developments will be presented and discussed in this chapter, developments that have eluded systematically STS, having rendered it as a field rather scientifically impotent. A number of interviews conducted by the author will be consulted, as well, to present some of the arguments formulated in this section. This thesis intends be the first of a projected series of social studies with an eye directed to scientific fields. The first field to be studied, the subject of the present thesis, is the field of mathematics, geographically located in the former Soviet Union, and today's Russian Federation. Archaeology will be just one among other disciplines employed in this study mostly as a theoretical guide, imposing an order, its own order, on the potential explanation narratives. And narrative, this latent

demon omnipresent in every molecule of social interaction, will be brought under extreme scrutiny, and the deeper relations of narration and mathematical proof will be examined. This new aspiring field will be housed under a most appropriate name for a deeply technical animal, such as *Homo sapiens: Anthropology of Engineering and Industrialisation*.

2.2 The Rise of Artefacts



Figure 2.1: Moscow State University - Main Building

Among the most interesting buildings in Moscow, offered to the Western tourist eye to feast upon, are the “Seven Sisters” of Moscow: these are seven buildings which were designed and built in the so-called Stalinist-Gothic style of architecture. The Main Building of the Moscow State University named after M.V. Lomonosov, located on the Sparrow Hills of Moscow, formerly known as Lenin Hills, is one of them (see Fig.2.1), and the statue of the Russian polymath Mikhail Vasilyevich Lomonosov is a permanent guard in the small park in front of the building. There are two more buildings on the left and right side of the statue and during the summer it is a fabulous place to visit and relax from the continuous buzz and noise of a very busy city. Walking in the park and with the building on the visual background

of a scientific discussion with a colleague is a very pleasing experience. The Main Building used to be for some time the tallest building in Europe, and it was build, along with its other six sisters under the leadership of Stalin. The interesting thing that concerns us in this thesis is that the Main Building houses the administration of the university, on the top floors along with the School of Mechanics and Mathematics, on the middle floors. The other two shorter buildings on the left and right sides of Lomonosov are the School of Physics and the School of Chemistry. The Main Building leaves a very strong stamp on the visitor's memory and lends Lomonosov University its distinct identity.

The location of the Mechanics-Mathematics School (or MekhMat in Russian), a few floors under the Dean's office, is indicative of the significance the Communist regime placed on the positive sciences: dominating over the humanities and the social sciences. After all, the nuclear and hydrogen bombs were built by mathematical and experimental physicists, at a time when the Cold War was about to start and rendered the Soviet Union a world class superpower. Once one gets inside, there are some chandeliers hanging from the ceiling on the ground floor, remnants of a revolutionary architectural propaganda, but still impressive enough. What is interesting to note is that as a workplace, the Main Building most probably is hardly noticed in its details by a member of staff, as it happens with most workplaces, unless an employee has a special interest in architecture. Despite its distinctiveness as an architectural artefact, the Main Building performs all the main functions a building performs: it houses the offices and lecture theatres of the departments, it provides protection from harsh weather conditions, provides with the equipment the teaching staff needs to execute their everyday tasks. A visitor still has to use the stairs or the lift to move between floors, still has to open doors to enter rooms, still has to grab chalks with his or her hands to write on the blackboard a mathematical formula. In other words, despite its "revolutionary" background, as a building it still performs the same fundamental functions a "capitalist" one would perform. Even an ancient Egyptian pyramid designer would probably have no problem in understanding the main functions of the Main Building of the Lomonosov University.

And some of these functions can be performed, for example, by a closed stadium. A car can be similarly used as a temporary housing facility, as well as a cave, maybe without the comforting company of a heating radiator during heavy winter time. And there is something more interesting than the Main Building of Lomonosov University, that was discovered during fieldwork in Russia, during interviews with Russian mathematicians, in particular, something that no other historical sources could reveal. When asked particular questions on some technical-mathematical topics related with their scientific work, the majority of the professors tried to explain these not only with words, but they were using hand movements, as if they were describing a building. When someone tries to describe a building to someone else, quite often he or she uses gestures to describe, among other things, what is on top, which room is next to which, or to draw an *imaginary* picture of the shape of the building. The cases where no hands were used were either because some object was held, or due to some social reasons, as, for example, when an interview had to be conducted in front of a small audience and there was some social stress involved. It is, for example, common knowledge that some shy people, as part of their effort to control their social image, try to control the movement of their whole body, hands and arms included, during social interactions depending on the comfort they feel at the time. Nevertheless, this line of *observable* behaviour showed that mathematics, especially its explanatory part in a classroom for example, is no simple fact, but more things are involved. And does this have to do with Russian mathematicians only? Well, actually no. As it was pointed out long time ago by a palaeoanthropologist,¹ in order to comprehend the emergence of stone tools, we have to see human language as an extension of artefact construction, and we have to see the writing of a scientific formula on a blackboard in the same fashion as the construction of a paleolithic stone tool. We have, as social scientists, although not customary for a social scientist, to go back in time a few million years ago, and become for a while palaeoanthropologists.

A very widely accepted definition of *tool use*, and, temporarily useful for our purposes, in general of artefact use, is:

¹Leroi-Gourhan in particular; later on, more will be mentioned on this.

The external employment of an unattached or *manipulable attached* environmental object to alter more efficiently the form, position, or condition of another object, another organism, or the user itself, when the user holds *and directly manipulates* the tool during *or prior to* use and is responsible for the proper and effective orientation of the tool. [37, p.5, italics in the original]

The italics denote the additional elements of the revised definition originally proposed in [36, p.10]. Although this definition of tool use is a general definition for *animal tool use*, and thus no particularity is attributed to the human species, a second definition, that of *tool manufacture*, is necessary, as well:

Tool manufacture is simply any **systematic structural** modification of an object or an existing tool so that the object serves, or serves more effectively, as a tool.

This definition is again a revised definition of Beck's original one proposed in [36]; the additional element, not contained in the original one is denoted in italics. Since "we now know that ... some apes and monkeys *sometimes* use tools to make tools," [37, p.11, my italics], the author of this thesis has added an extra element to the definition, denoted in typewriter fonts, to lend a more Homo-sapiens-centric orientation. This definition, as happens with every definition, is not perfect, and for more information the interested reader can look for a history of definitions of tool use and alternative definitions in [42] and [37, p.1-23]. The key word of the second definition that concerns us in this thesis is the *modification* aspect: modification of an object necessarily implies *intentional* modification, that is, modification with a plan, with a picture of an approaching and desired future. This intentional modification of "inorganic" material artificially detached from the surrounding ecosystem requires an unusually higher level of animal intelligence.

The oldest of *archaeological industries*,² that is, archaeological sites with a high number of a certain type of artefacts scattered over an area of land,

²Also sometimes called *techno-cultural complexes* or simply *technocomplexes*.

usually stone tools, are the *Oldowan industries*. They they were first discovered by the British archaeologists Louis and Mary Leaky back in the 1930s, and have been named after the discovery site of Olduvai Gorge in Tanzania. The oldest such industry is located in Gona, Afar, Ethiopia, and has been dated at around 2.6 mya (million years ago) [407], without excluding the possibility that the actual use of tools started much earlier [70]. The first tool users seem to be members of the species *Australopithecus garhi*, in the genus *Australopithecus*, a species which appears to be the evolutionary missing link between the genus *Australopithecus* and the genus *Homo* [23]. Main users of Oldowan stone tools were the species *Homo habilis* and *Homo ergaster* [386]. A later species, *Homo erectus*, seems to have picked up on the Oldowan technical culture and then spread outside of Africa, before 1.8 mya. *Homo erectus* Oldowan industries have been found in Java [428], dating at 1.8 mya, in China, dating at 1.66 mya. Remainings of Oldowan technology, but no fossil skeletons, have also been found in Spain, dating at 0.78-1.77 mya [350]. Many more other Oldowan industries have been discovered in Africa, Asia, and Europe. There are many other *Homo* species and subspecies identified by archaeologists,³ but we are interested as social scientists, at this point, more in the artefacts, rather than in the biological history of their users.

Oldowan stone tools, also sometimes called *pebble tools*, have been called *Mode 1 tools*. These are the oldest and simplest tools that have been found (see Fig. 2.2⁴). A major function of stone tools was to process large animals [70]. In order, most probably to eat, the first tool users had to cut the raw meat, which is very difficult to cut without a knife. Carnivorous animals, unlike all the *Homo* species are very well equipped with teeth specialised for this function. Stone tools were also used to break the animal large bones to gain access to the marrow inside. These are known due to the fact that when stone tools are used for this functions they leave distinct traces on the bones, traces which have been found on fossilised bones. Similar traces have been found on fossilised wood, which led archaeologists to conclude that, quite probably, wooden tools were used, as well, in everyday routines. Different

³For more on taxonomy see, for example, [463].

⁴Downloaded from Wikipedia.



Figure 2.2: Oldowan chopper: side views and a view from above.

tools might have been used for digging the ground or for processing useful plants. The archaeological focus has been also on the hand anatomy of the early *Homo* species, when compared with modern nonhuman primates, such as the great apes or capuchin monkeys (see Fig. 2.3⁵). Although the use and manufacture of tools is indeed related to the anatomy of the hand, the hand itself is related to other uses, as well. It is important to note, at this point, that the hand is the mediator between the tool and the brain, and therefore, as we will see later on in this chapter, it is not only the anatomy important, but also the neural coordination between the executive brain and the executing hand.

Mode 1 tools are called also pebble tools because they have not been processed too much. They are still visibly stones, like pebbles. The chopper of Fig. 2.2, for example, is still a stone on its lower half. The lower half is the

⁵Figure and text taken from [70].

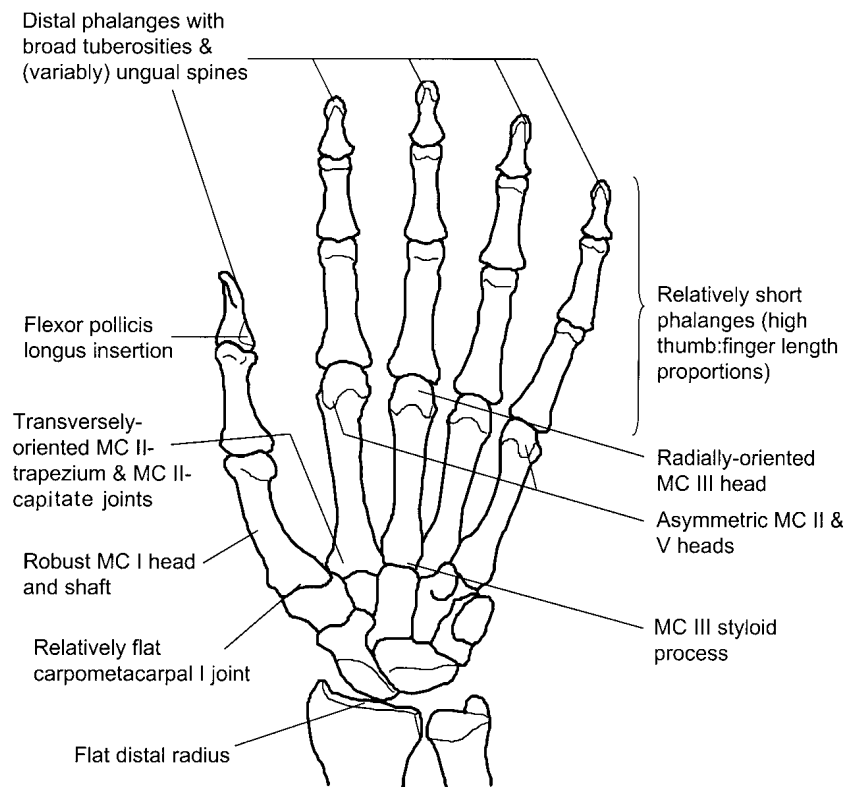


Figure 2.3: “The great apes have relatively opposable thumbs and thus are able to make pad-to-pad contact between the index finger and thumb, although not as completely as humans can. They are also capable of a wide range of manipulative behaviours. Capuchins have a pseudo-opposable thumb and are capable of both power and precision grips.” MC = metacarpal [70, p.240].

part that is held by the hand in order to chop the animal’s meat. What is interesting though is the fact that this stone tool has been constructed by using other stone tools: a large *hammerstone* held by hand has been used to strike off flakes from the original stone intended to be the chopper, which is the *core* (see Fig. 2.4⁶). The possibility also of having used a large flat-like very hard stone as anvil on which the core was put cannot be excluded. These processes have been conjectured by experimental archaeologists, who themselves have tried to (re)construct stone tools (see for example [449]). What is important to remark is that the first production line, although very

⁶Downloaded from Wikipedia.

primitive, appeared: tools being used to produce other tools on a *systematic*, that is on a planned and repetitive, basis, not just out of problem-solving abilities circumstantially called forth. In other words the oldest surviving evidence of human culture is *industrial* culture. Animals, and nonhuman primates in particular, are circumstantial tool users, but the species of the genus *Homo* started as *engineers*.



Figure 2.4: Using a hammerstone, on the right, against a stone core, on the left.

The next industries, in terms of time and technological development, are the *Acheulean industries*. They were named after Saint-Acheul, a suburb of the city Amiens, in northern France, where they were first discovered.⁷ These industries are most commonly related to the *Homo erectus*, that is, “Human the standing” in Latin, species [284, p.166-169]. The upright position was instrumental in the development of tool-using biological mechanisms, as we will see later on in the text. Acheulian artefacts seem to have been firstly used 2.8 mya [280] in certain African locations, which means that the Oldowan and Acheulean technologies were not necessarily mutually exclusive technologies. Probably “the Acheulian [industry] was either imported from another location yet to be identified or originated from Oldowan” tool-users in the vicinity of the industry at issue [280]. *Homo erectus* migrated from Africa to Eurasia as early as 1.7-1.6 mya, most probably in three waves

⁷Actually Acheulean tools had been discovered much earlier; see [184, p.55-56].

[26]. Possible reasons for these migrations seem to have been environmental changes, population pressures or expansion of predatory range. *Homo erectus* seems, additionally, to have been the earliest *Homo* species to have used fire [46], maybe circumstantially, with important implications on the dietary regimes of the individuals, another development which has been recently conjectured to be an important factor in the increase of the brain volume of later *Homo* species.⁸ It has been suggested that *Homo erectus* could travel over the open sea on rafts [169], and it is thought to have been the first *Homo* species to be organised in hunter-gatherer band societies [56, p.198].



Figure 2.5: Construction of an Oldowan tool (top) and an Acheulean one (bottom).

The Acheulean tools, or *Mode 2 tools*, are more widely known to the general public: these are the *hand-axes*, or *bifaces*, in the technical parlance of archaeology. They were more elaborate in their construction, and the end result was a two-sided stone tool, that is, a biface (see Fig. 2.5⁹). When compared with Oldowan and post-Acheulean industries, Acheulean industries is actually an understudied subject in archaeology [88] (but see [409]). Without special training as an archaeologist, even a lay person cannot avoid noticing the increased elaboration of a hand-axe when compared to a pebble tool.

⁸More on this later on.

⁹Taken from [94].

Both sides of the stone have been processed, or reprocessed for a second time [322], and even for a modern human it is necessary to spend some time in training to achieve such a sophistication in manual skill:

The production of bifacial tools is a complex system integrating long-term planning (using both working memory and planning memory) and its step-by-step implementation to obtain the desired end-product. This requires knowledge, experience, communication and the flexibility to change the procedure according to the circumstances [177, p.1044].

In spite, though, of the high sophistication of the handaxes, there is a certain uniformity in shape, which led some archaeologists to assume, the existence of a mental template common to all *Homo erectus* [466]. In other words, the handaxe makers had a certain *aesthetic intention* in mind when constructing a stone tool (see Fig. 2.7¹⁰). The variability of the available stone material resources over a large number of Acheulean industries, for example, did not seem to affect the variability of the handaxe shapes [410]. Other researchers, on the contrary, have claimed that this observed common shape reflects its better functionality and the processing construction sequences, rather than a common aesthetic mental template [322]. What unifies these seemingly divergent scientific judgements, is that *Homo erectus* had indeed some kind of *intention* in mind, whether this was an elevated aesthetic one or a more fundamental functional one.

The emergence of a new discipline studying the interaction between the brain of the *Homo* species and material culture, called *Neuroarchaeology*, has led to some new scientific explanation narratives within the field of archaeology. From experimental archaeology we now know that “[t]he circumstances of knapping involve[d] complex feedback between limbs, objects, the visual sub-system, and the acoustic sub-system (because there are distinctive sounds associated with the successful removal of a flake)” [112] (see Fig. 2.6¹¹). Since the act of knapping is actually a series of acts, until the stone tool has the

¹⁰Figure and text taken from [467].

¹¹Fig. 2.6 is taken from [422].

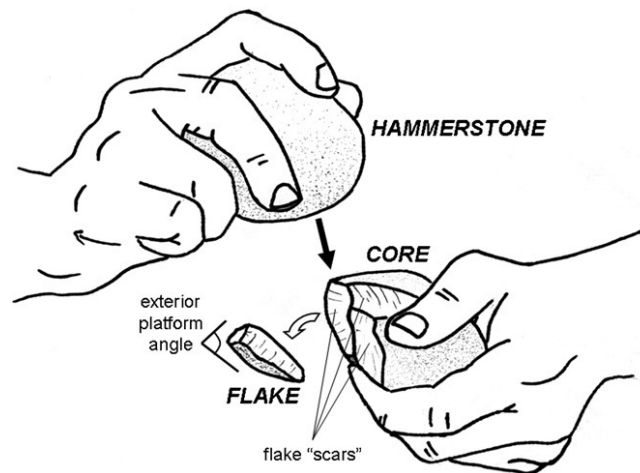


Figure 2.6: “Knapping as *enactive cognitive prosthesis* (ECP). The knapper first thinks through, with and about the stone (as for example in the case of Oldowan tool-making) before developing a meta-perspective that enables thinking about thinking (for instance as evidenced in the case of elaborate late Acheulean technologies and the manufacture of composite tools)” [306, my emphasis].

desired shape, the whole process of knapping plus past stone knappings actually become part of the tool maker’s cognition informing future knappings. In other words, the act of stone knapping becomes an *enactive cognitive prosthesis* (see Fig. 2.6). As Dennett had quite early noticed,

tool use is a two-way sign of intelligence; not only does it require intelligence to recognize and maintain a tool (let alone fabricate one), but a tool *confers* intelligence on those lucky enough to be given one. The better designed the tool (the more information there is embedded in its fabrication), the more potential intelligence it *confers* on its user [120, p.99-100, my emphasis].

The act of knapping, in other words, participates in the construction of a stone tool, and the stone tool being constructed is actually *lending* intelligence to its user, who is no more a user, but simply an operator in the whole process. There is a marked similarity at this point between stone knapping and solving mathematical equations, as both are actually enactive cognitive

prostheses, but still the tool maker is not yet, at this point of our scientific narrative, a fully fledged *Homo sapiens*. Stone tool knapping actually marks the beginning of discharging the human species of its societal duties and highlights the promotion of a new actor in its place as a social agent: matter.¹²

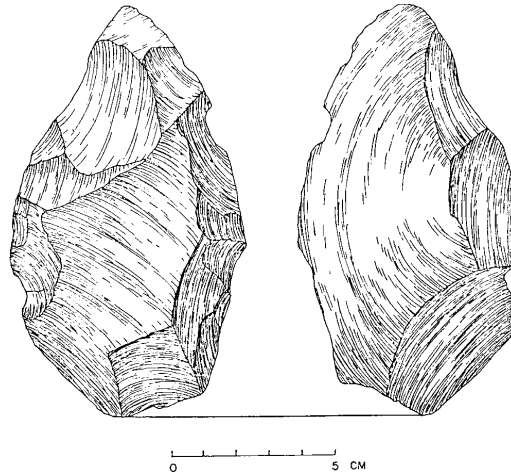


Figure 2.7: “1.4 million year-old handaxe from West Natron, Tanzania. The artifact has a ‘global’ bilateral symmetry. The lateral edges mirror one another in quality of shape, but are not congruent.” [467]

There is another aspect of the Acheulean industries not so well researched in archaeology based on the concept of *autopoiesis*, that is, how stone tools, although inorganic, produce and reproduce themselves. In 1959 the entomologist Grassé introduced the concept of *stigmergy*¹³ in order to explain how the collective behaviour of ants emerges as the result of the individual behaviour of each ant:

The basic principle of stigmergy is extremely simple: Traces left [e.g. pheromones] and modifications made by individuals in their environment may feed back on them. The colony records its activity in part in the physical environment and uses this record to

¹²For more on the extended mind theories and the material agency approach see [95, 249, 307, 344, 400, 446].

¹³From the Greek words *stigma* = sting, and *ergon* = work.

organize collective behavior ... Stigmergy also solves the coordination paradox: Individuals do interact to achieve coordination but they interact indirectly, so that each insect taken separately does not seem to be involved in a coordinated, collective behavior [437, p.111].

As the individual ants, for example, walk on the ground in search of food, they leave traces of pheromones, which other individuals from the same colony can smell. When food is found the other members of the colony simply follow the traces, or lay down their own. In this way this network of pheromone traces act as an *external memory* of the colony, with respect to its every day running tasks. Moreover the individuals are in position to differentiate the pheromones of their colony from that of other colonies, and from other environmental smells, and also can optimise paths to food sources (see Fig. 2.8¹⁴). In other words, the pheromone traces *confer meaning* to the individuals, and they *mediate* between the colony and each individual. Following up on the last paragraph, we can now say that the pheromones have become external agents both to the colony and to each individual ant. Stigmergic behaviour has been found to organise the everyday work of builders conducted on construction sites [92]. Besides the study of social insects, stigmergy is being used extensively as *optimisation algorithm* in computer programming [128], in artificial intelligence analysing *swarm behaviour* [55], and, recently, in social network theory modelling the *emergence of leadership* [286], and as an explanation narrative of *spontaneous economic order* [129].

Let us now return to the problem of the uniformity of shape in Acheulean handaxes. It has been found that the ratios of a handaxe's dimensions,¹⁵ such as that of its length over its width, are dominated roughly by the "Golden Section", usually denoted by the Greek letter ϕ [phi] [368]. The Golden Section has been related quite often to aesthetics and beauty [294]. Moreover, it has been studied and used in Euclidean and post-Euclidean geometry [215]. The stone tool makers, obviously, did not know any mathematics or any Euclidean geometry. This led many archaeologists to claim that *Homo erectus*

¹⁴Text and graphics taken from [55, p.47].

¹⁵These are called *allometric relationships* in archaeology.

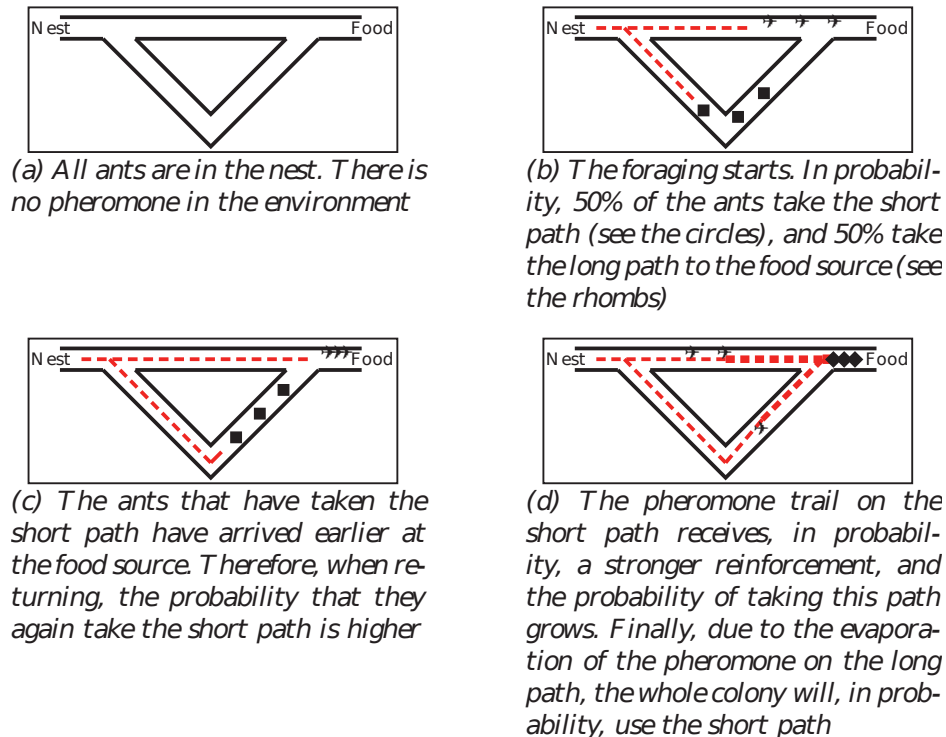


Figure 2.8: An experimental setting that demonstrates the shortest path finding capability of ant colonies. Between the ants' nest and the only food source exist two paths of different lengths. In the four graphics, the pheromone trails are shown as dashed lines whose thickness indicates the trails' strength [55, p.47]

had a sense or a feeling of visual symmetry. Even by looking today at Fig. 2.7 any reader can identify a certain symmetry on the biface:

Essentially stone artefacts must have form - or we could not recognize them. [...] If it was the same as natural form - we would not know that they were artefacts. [...] [Moreover a]rbitrary form need not be diagnostic of humans - it is diagnostic [in general] of [animal] artefacts [179, p.6-7].

In other words the handaxes can *still talk* to the modern *Homo sapiens*, that is, their original *Homo erectus* meaning is still perceivable by modern humans. And the meaning of a hand axe is not only its form, but also its *potential* technical usefulness or uselessness: it can be picked up, thrown

away or hidden away. It is interesting now to note, that whatever the personal meaning the stone tools had for each *Homo erectus*, it has begun to emerge that the discard of hand axes is actually structured and patterned according to stigmergic principles, when considering the whole land areas, that is, where the *Homo erectus* moved, lived, and hunted [368]. If Acheulean stone tools, therefore, were “discarded contextually within stigmergic systems, [they could] fulfil key semiotic roles beyond simple technological function” [368, p. 53]. In other words the smell of the organising feromones in ant colonies, might have become visual symmetry, or functionality, in the case of *Homo erectus* band societies, which, unlike the organic ant feromones, could persist over time. One could be led, as well, to claim that a major part of modern experimental archaeology, that is, the part that tries to reconstruct stone tools, along with many amateur enthusiasts over stone tool manual (re)construction, are actually the products of 1 million years old reactivated, previously latent, nonhuman material agency, whose original purpose was to reproduce itself by means of the *Homo erectus* species.

The *Mousterian* industries, which are called *Mode 3* stone tools, are associated with *Homo neanderthalensis*, that is, the Neandertals, which as a fully deveveloped species appeared 150,000 years ago [381, p.379], although skeletal remains with proto-Neandertal traits date back to 350,000 to 500,000 years ago [49]. The name comes from the Neander valley in Germany,¹⁶ where the first fossilised remainings were found. There has been an ongoing debate on whether the Neandertals were a species of their own, or a subspecies of *Homo sapiens* [434], since there have been findings of Cro-Magnons, that is, early modern humans, with “Neanderthal traits” [131]. Recent genetic evidence points to the direction of possible interbreeding between the Neandertals and the *sapiens*; however other possible scenarios cannot be ruled out [185]. Neandertal remains have been found in western Europe, the Balkan peninsula, middle East and central Asia. They formed hunter-gatherer band societies, and their prey was, quite often, large animals, a very strong sign of cooperative behaviour and organisation [469, p.22-48]. They seem to have been using a form of proto-language and they had developed symbolic percep-

¹⁶Written as *Neanderthal* in old German spelling, and *Neandertal* in modern German.

tion: the gene FOXP2, identified with an important role in human language, has been found in Neandertal gene, as well [256]; moreover, a Neandertal hyoid bone, which is involved in the articulation of human language sounds, has been discovered in Israel [20]. They buried their dead, but there is no evidence so far as to the performance of a burial ritual. It is also believed that they practised cannibalism [445]. While some of them lived in caves, some others have been found to use mammoth bones to built shelters [117].¹⁷ The Neandertals disappeared around 30,000 years ago [469, p.1]. It is known that they coexisted with *Homo sapiens* for a few millennia [323]. Various theories have been proposed as to their extinction: excessive attacks on very dangerous animals [419], repeated attacks from *Homo sapiens* groups [316], spongiform encephalopathy due to cannibalism [90], and inability to adapt to climate change [411].

Mode 3 tools, as one would expect, are more complicated and demand extreme skill and experience. Much research focus has been lately on the *chaîne opératoire*, or *operational sequence*, in tool construction, according to which “an assemblage of lithics is not a random but a methodically interconnected association of artifacts” [27]. In other words, the assemblages are never random, and their artefacts were scattered according to a pattern, which depended upon the every day life of Neandertals. Archaeologists have, for example, identified the *Levallois technique* in, mainly Neandertal, stone tool construction¹⁸ which consists of three steps:¹⁹

1. “The first prepares a core with two distinct but related surfaces, one, a more convex platform surface that will include the striking platform, and a second flatter production surface from which the blank or blanks will be removed.”
2. “The second step prepares the striking platform itself in relation to the axis of the intended blank.”

¹⁷See also [469] for a recent, updated, and fascinating account of Neandertal life from many perspectives.

¹⁸The construction methods of stone tools are classified in general under the rubric *core reduction techniques*.

¹⁹Taken from [468].

3. “The third step is the removal, by hard hammer, of the blank or blanks. The first two steps each encompasses its own sequence of action, as does the final step if multiple blanks are removed.”

The main change we can notice in the construction of stone tools are the steps involved: not the steps themselves as to what is constructed, or how the construction is being done, but the fact that there are discrete steps involved. In other words there is now an obvious planning of the whole procedure, rather than simply knapping flint. And, since future material movement can now be planned, all the artefacts involved in the construction process *acquire imaginary locomotion*, a fundamental property of the ability of modern humans to narrate. There are Mode 4 and Mode 5 tools, identified by archaeologists, whose appearance and construction coincides with the appearance of modern humans. At this point of our narrative, though, the archaeological evidence is quite eloquent on the emerging relationship between artefacts, human cognition and societal organisation. Archaeology, and cognitive archaeology in particular, has actually become the continuation of ethnography by means of materiality, an approach which will be useful, later on, not in its details as a particular archaeological method, but as a stepping stone for a new method more appropriate for modern complex societal (scientific) industries: an *archaeology of modernity*. But our story is not over yet.

2.3 The Revolution and Convolution of Evolution

In February 2011, an important article was published on the Journal of Archaeological Science, which went largely unnoticed by the STS community [241]. The authors conducted a very interesting experiment. A group of 60 people were divided into two groups: one group was given steel blades, while the other was given flint flakes produced by the scientists themselves. The participants were instructed to cut a rope with the cutting tool each one was

given, using their dominant hand (see Fig. 2.9²⁰). None of the participants any experience in flint knapping, nor any formal education on this subject prior to the experiment. Certain biometric characteristics were measured, handsize and grip strength in particular, and then certain measures of cutting efficiency were set up: number of “total cutting strokes required,” and “total time taken” until the task was accomplished. Since the hand size and the grip strength are highly correlated [97], and since the biometric measurement that survives through time and remains measurable by archaeologists is only the hand size, the statistical analyses conducted after the experiment were focused on the hand size measurements and their relation to the cutting efficiency of the tools. Without getting into too many technical details, the results “demonstrated a statistically significant relationship between the biometric parameter of handsize [sic] and the efficiency of cutting ability as measured by two different measures of efficiency” (p. 1667). Considering the fossil findings of early hominids, such as *Homo habilis* and *Homo erectus* we know that the hand, along with the whole human body, has evolved. This evolution has happened under the strong influence of environmental factors. In other words, as the hominids started to branch out from the common ancestors of all modern primates, at some point the correlation between hand and tools started to increased, while in the rest of primate evolutionary branches, this correlation remained very low. This statistical significance discovered between the hand size and the cutting efficiency, actually shows that the use of stone cutting tools was an environmental contributing factor to the evolutionary transformation of the hominid hand. The current archaeological evidence, therefore, suggests that stone technology was indeed a contributing factor to *human speciation*, that is, to the emergence of *Homo sapiens* as the latest hominid species. Another equally interesting conclusion of this study was that the type of the cutting tool, that is, flint flakes versus steel blades, did not actually affect its efficiency, reiterating thus “the efficiency of simple stone flakes as cutting tools (at least in tasks of short duration), even compared to industrially produced steel cutting blades” (p. 1668).

The previous narrative is typical of evolutionary anthropology and sociobi-

²⁰Both picture and text taken from [241].



Figure 2.9: Example of flint flake cutting through hessian rope [241, p. 1666].

ology, disciplines which are necessary for the production of a scientific narrative suitable for STS. The fundamental textbook of this line of thinking is Darwin's famous, and infamous, "The Origin of the Species" [110]. It is a well established fact today, that Darwin's theory met with fierce opposition, not to mention his own initial hesitance to publish his ideas [122]. A few decades later another similar book by Wilson [460] merging biological and sociological narratives popularised the term "sociobiology" causing a lot of controversy, even within the discipline of biology (see [6], and for a more extended account see [406]). A more recent anti-Darwinian campaign is the neo-creationist religious movement of *intelligent design* [19]. These heated debates indicate the continuous prevalence of ideology over science, in our societies in general, an ideology which has not left unscathed the social sciences, and STS in particular. The purpose of this section is to introduce briefly this evolutionary-anthropological argumentation and explain some of its controversial arguments, which, according to the author, are mostly the result of misunderstanding, rather than the result of ideological-religious differences. In fact, any toddler can see that a cat, a dog, and a human have four legs, spine, head, and fingers, toes, and nails: that has been obviously not accidental.

Probably the most heated debates on evolutionary biology have circled around

the concepts of *natural selection*, and *adaptation*. When Darwin talked about natural selection, he used this expression as opposed to artificial selection, which is what we would today call *selective breeding*. We could also alternatively call it spontaneous selection. Probably the choice of the word selection has not been quite successful, since it implies the existence of a concrete selecting agent, but it has become standard in biology. An *adaptation* could be called “a characteristic that enhances the survival or reproduction of organisms that bear it, relative to alternative character states” [161, p. 279]. The first thing that has to be mentioned is that the adaptation characteristics of a species have to be identified by scientists themselves first, then proposed as such to research peers, and then be established as standard by the scientific consensus reached after a period of time has lapsed. In other words, it totally depends on the limitations of the human intelligence and the visionary ability of the biological scientific community, as members of a biological species. The same holds, as well, for the scientific consensus of the mathematical community. “ ‘[N]atural selection’ [on the other hand] is not synonymous with ‘evolution’ ” (p. 283), since evolution can be the result of other means, such as *genetic drift*, a way of spontaneous genetic variation. Natural selection is very closely related with *fitness*, that is, the *reproductive success* of a species as a group, which can be measured by the survival probability of the subsequent generations of a species, or the average number of offspring over a number of successive generations. There are levels of selection, such as on the genetic level, cellular level, individual level, or group level (see the discussion in [314]). By far the most discussed in the scientific literature are the levels of the individual, *phenotypic variation*, and that of the gene, *genotypic variation*. The most cited teaching example of natural selection in the literature is the evolution of peppered moth [305]. About a hundred years ago the peppered moths had light coloration, which helped camouflage themselves against their predators by sitting on light-coloured trees in the English countryside. With the advent of industrial revolution, many trees in urban industrial areas became covered in soot. As a consequence, the population of the light-coloured moths begun to diminish, resulting in their disappearance from these areas. At the same time a new population of dark-coloured moths appeared which could camouflage themselves easier in a dark and polluted

environment (see Fig.2.10 ²¹).



Figure 2.10: *Typica* and *carbonaria* morphs [i.e. phenotypes] resting on the same tree. The light-colored *typica* (below the bark's scar) is nearly invisible on this pollution-free tree, camouflaging it from predators.

A more recently established instance of human natural selection was the evolution of lactase persistence, that is the ability of adult humans to digest the milk sugar lactose.²² Digesting lactose is of vital importance to an infant mammal, but later on, after weaning this Mendelian, that is, inherited, trait declines [426]. Intolerance to lactose can be dangerous, and can lead to ““abdominal pain, diarrhea, nausea, flatulence, and/or bloating after the ingestion of lactose or lactose containing food substances” [216, p. 1280]. Lactose persistence, therefore, would be an extremely important evolutionary advantage, in the first agricultural human societies, when cattle were first domesticated, and the cultural practice of adult milk consumption started.

²¹Photo and text downloaded from Wikipedia.

²²Lactase is an enzyme whose (only) function is to break down the molecule of lactose during metabolism into glucose and galactose molecules, which are easily digestible [426].

Modern humans are actually the only mammals today, whose adult individuals can digest milk, that is, they are lactase persistent. Lactase persistence is very frequent among northern Europeans, less frequent in south European and middle eastern populations, and diminishes significantly in nonpastoral populations of Asia and Africa; quite interestingly it is a trait of very high frequency in pastoral populations of Africa [64]. In a study conducted in 2006, various African pastoral populations were checked as to the corresponding gene mutations, that kept the lactase persistence gene “switched on,” ([438]). Three more mutations were discovered which were independent of the European one, and of each other. Moreover, these mutations were more recent than the European. The advantage that these mutations could confer on the individuals would be tremendous, since, besides the steady consumption of milk, the water contained would be also useful during periods of drought. These findings were actually “one of the strongest genetic signatures of natural selection yet reported in humans,” (p. 32). We can see, with this instance of natural selection, individuals without this mutation would be actually deselected. In other words, natural selection is actually natural deselection: species without appropriate survival traits, simply perished. In a book published in 2009 [465], the cooking hypothesis was formulated and presented: the human brain needs a lot of energy, which cannot be acquired simply by eating raw food; cooked food, on the contrary, offers lots of energy in smaller quantities, more digestible, less time in cutting meat, and less time in foraging. Cooked food, in other words, offered, according to this view, new survival traits to the hominids one million years ago. Although this view was initially criticised as simply a theory due to lack of evidence, last year archaeological evidence was discovered, confirming that *Homo erectus* had already acquired control of fire 1 mya [46]. This was one of the, still, very rare cases in evolutionary biology, in which a theory anticipated evidence.

During the last decade, more and more biologists are turning to a new converging research direction: the coevolution of culture and human biology. The problem of culture, that is, of the definition of culture, has always hindered scientific consensus among the social scientists: there exist practically as many definitions as many researchers. In the biological sciences, on the

contrary, “[r]ather than attempting to describe the entire culture of a society, culture is broken down into specific traits (for instance, milk users or non-users, or consumption of a starch-rich or starch-poor diet), which allows their frequencies to be tracked mathematically” [267, p. 138]. Moreover, this definition allows culture to be transmitted through (observational) social learning, and, therefore, animal behaviour can be included, as well (for more on animal culture see [265]). There are two areas studying gene-culture co-evolution: one is *gene-culture co-evolutionary theory* [63, 87, 142, 147], which employs mathematical models of population genetics, and studies cultural and genetic processes in the course of evolutionary time; the other is *niche-construction theory* [262, 266, 346]. Niche-construction theory explores the impact of an environment modified by a species, such as nest building, on its evolutionary history. In human societies, this constructed niche is evidently material culture. Actually, it is now openly asserted that “gene-culture co-evolution could be the dominant mode of human evolution” [267, p. 137] (see also [139, 387, 264]). The population genetics models assume the existence of a hypothesized gene function as observed in phenotype, and study their dynamics with cultural behaviours. Such an evolutionary model was proposed in 1986 [12] to explain adult lactase persistence which, as we saw earlier, had its conjectured gene confirmed some 25 years later by genetic studies on populations. There are now many available mathematical models of gene-culture co-evolution which connect certain cultural traits with a conjectured gene phenotype. There have been also many genes identified whose function was subjected to cultural environmental pressures (for more on the mathematical models and the identified genes see [267] and the references therein).

Turning now to the neurobiological sciences, the effects of biological evolution are becoming more and more visible. The human brain now is being examined with respect to its functionality in comparison to other mammal, in general, and primate, in particular, brains. But before going into details, a brief review of the human brain anatomy is necessary. The major part of the brain consists of “[t]he *cerebral hemispheres* [which in their turn] consist of a heavily wrinkled outer layer - the *cerebral cortex* - and three deep-lying struc-

tures: the *basal ganglia*, the *hippocampus*, and the *amygdaloid nuclei*. The basal ganglia participate in regulating motor performance; the hippocampus is involved with aspects of memory storage; and the amygdaloid nuclei coordinate the autonomic and endocrine responses of emotional states. The cerebral cortex is divided into four lobes: frontal, parietal, temporal, and occipital” [232, p.9, italics in the original] (see Fig.2.11)²³. The cortex is fundamental in organising most of the major cognitive functions, such as awareness, language, thought, attention and memory. The first six outermost layers of the cortex comprise the *neocortex*. The neocortex appeared later in the history of evolution, and it is a distinctive feature of mammalian brains, although there is some ongoing debate on terminology as to whether this six-layer cell structure in mammals other than primates should be called neocortex, as well, or not (see for example [229, 380]). Each of these six layers is numbered in Roman numbers from I to VI, the first one, i.e. layer I, being the outermost, and the last one, i.e. layer VI, being the innermost. In larger mammals, such as chimpanzees, the neocortex consists of *sulci*²⁴, i.e. grooves, and *gyri*²⁵, i.e. wrinkles, while in smaller mammals, such as mice, it is almost smooth-surfaced [219, p.147].

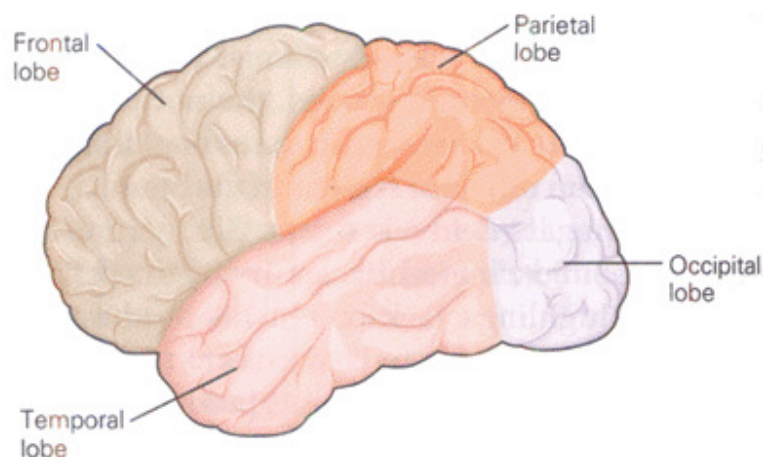


Figure 2.11: The four lobes of the cerebral cortex.

²³ Figure and text taken from [232, p.9].

²⁴ Singular *sulcus*.

²⁵ Singular *gyrus*.

It was formerly assumed that the various cortical areas are specialised in certain cognitive abilities. Broca's area, for example, located on the left posterior inferior frontal gyrus of the brain, that is, on the back lower frontal left lobe, used to be assumed to be involved in language generation [69]. This view of the specialisation of brain areas has become known as the *modularity of the mind* [151]. Nowadays, though, "[t]here is increasing evidence that language regions in the brain - even classic Broca's area - are not specific to language, but rather involve more reductionist processes that give rise to language as well as nonlinguistic functions" [57, p.152-153]. Tool use, our major concern here, has been found to be involved as well in the function of language brain areas [217, 423]. Since all the primates have a common ancestor, in terms of biological evolution, they have developed *functionally homologous* regions on the brain, that is regions which perform the same or similar functions and are usually anatomically and physiologically similar, although anatomy and similarity are not always good advisors on functional homology (see, for example [425]). Five-finger hands with a thumb, for example, are homologous limbs on all primates, and grasping is an homologous function. It has been repeatedly demonstrated that areas on the brain of other primate species functionally homologous to the language areas of the human brain are involved as well in tool use [217]. This has given a new rise to the formerly proposed, but later largely ignored, evolutionary mechanism of *exaptation* [178], that is, the functional reassignment of an evolutionary biological characteristic. One commonly cited example of exaptation is that of feathers in birds: while feathers were developed originally as a mechanism of temperature regulation in *Archeopteryx*, a transitional genus between the late feathered dinosaurs and the early modern birds, and whose members were too heavy to fly [471], "[l]ater selection for changes in skeletal features and feathers, resulted in the evolution of flight" [178, p.7]. These findings, therefore, in the functional activity overlap of brain regions, "supports the gradual view that the neural correlates of sequentially organized behaviour, exemplified by tool use, were already present in a rudimentary form in our last common ancestors with primates, and were later exapted to support language in humans" [217, p.1381].

In 1991 a scientific report was sent to *Nature*, but the editors rejected the “paper for its ‘lack of general interest’ and [they] suggested publication in a specialized journal” [393, p.223]. After some discussions with editors on other journals, the article was finally published after a few months [124], marking the appearance of *mirror neurons* in the scientific literature; two types of visuomotor neurons had been discovered: “canonical neurons, which respond[ed] to the presentation of an object, and mirror neurons, which respond[ed] when the monkey sees object-directed action” [392, p.170]. The object alone, i.e. the sight of it, did not trigger the neuron; an interaction between the object and the hand and/or the mouth triggered it. Two types have been identified in animals: *ingestive mirror neurons*, and *communicative ones*, involving, among others, gestures. Mirror neurons have not been directly observed in humans, for the very simple reason that on ethical grounds researchers cannot implant an electrode directly on a human brain, but there is more than enough indirect evidence, pointing to that direction [272]. What is, indeed, intriguing is that animal mirror neurons are located on brain areas, e.g. area F5 on monkey brain, which are homologous to Broca’s area in human brains [430], rendering the modularity approach in language production obsolete for good [362]. All these have led to the *Mirror System Hypothesis*, according to which “this primitive action-matching system underwent successive evolutionary modifications to support imitation, pantomime, manual ‘protosign’ and ultimately vocal language, thus providing a neural underpinning for ‘gestural hypotheses’ of language origins” [423, p.76]. Although the Mirror System Hypothesis has been viewed as an explanation panacea for many human culture domains, such as empathy, understanding intentions, self-awareness, language, imitation, even gender differences, many gray areas remain, as well as some skepticism among the scientific community (for more on these see [14, 359], and for some criticism see the peer commentary following the main article in [13]). In the framework of the present thesis, the mirror neurons will be considered as the first evolutionary pangs of imagination, as the lowest level building blocks of vicarious social action.

Going back, now, to where we started, that is the stone tools, we could attempt to give a definition of human culture from the point of view of an

external observer: “[...] culture, in addition to being ‘... that complex whole ... shared by man as a member of society,’ is also the *imposition of arbitrary form upon the environment*” [220, p. 395, his emphasis], and form means *material form*. In the case of language form refers to: (a) its *phonology*, or sound production, which in the case of tools “involves such units as striking a flake; (b) its *grammar*,²⁶ that is the ordering of (material) sound units to a meaningful whole, which in the case of tools can be “the concatenation of smaller unit operations that produces the tool;” and (c) its *semantics*, or meaning, which in the case of stone tools can be “the use of the tool as finished product, and the meaning of each unit action as an outcome of the preceding one and as preparation for the next.” One more element could also be added: (d) its *pragmatics*, that is its meaning as *embedded* in social activity. Human culture, in other words, cannot be symbolic, only material: “[p]roperly speaking there are not signs, but only *sign-functions*. ... A sign-function is realized when two *functives* ([material] expression and [semantic] content) enter into a mutual correlation” [136, p. 49, his emphasis]. It has been demonstrated by Greenfield that primate infants, humans included, adopt certain action patterns in combining linguistic utterances and motor activities; these patterns have been called “grammars of action” [186, 187]. These action grammars have been found to be similar among primates infants, and begin to diverge later on with age, and it has been concluded that these are similar due to the existence of a common evolutionary ancestor. The research emphasis was more on utterances, rather than on motor activities, “because there is no fossil record of behavior” [186, p. 545]. Modelling after these studies, some archaeologists have proposed “a model of the ‘design space’ of knapping - the essential actions of stoneworking - in terms compatible with Greenfield’s model” [329, p. 14]. In conclusion, although there are many gaps in the archaeological record, and many objections and doubts exist as to the precise explanation narratives proposed, there is a widespread general belief in the archaeological community, that stone tools and language are actually more intimately related, than it was generally believed 20 or 30 years ago (see for example [423], and the references therein). The present thesis follows up on this educated belief, in order to formulate a sociological research model for

²⁶Or rather *syntax*.

the mathematical sciences.

2.4 O Mind, Where Art Thou?

It is important to note, at this point, that the social-constructivist approach in the social sciences, although in high vogue, is not fully adequate for the purposes of this thesis. Studying science in general, and mathematics in particular, from the singular point of view of social construction, normally would not necessitate a literature review of archaeological and biological evidence. It seems, though, to the author that there is, in the first place, a certain “dialectic between social construction and neurological foundations” [287, p.112] which is too important to be ignored. This legitimates partly the adoption of the evolutionary scientific narrative so far, and in the following sections as well. The attempt is to redefine scientific rationality based on every appropriate evidence available, producing thus a method of study appropriate for the ethnographic disciplines. The turn to the interaction between material culture and human consciousness is a further attempt to lend cultural anthropology and linguistics, two of the traditional anthropological disciplines, the status of positive sciences, a status which, so far, is being enjoyed only by archaeology and physical anthropology, the other two traditional anthropological disciplines. The main problem lies in the Cartesian brain/mind duality:

If we are speaking of evolution, we are speaking - essentially - of the human body, our physical, material make-up of bones, blood, tissue, brain matter. By contrast, mind is a projection, an abstraction; it cannot be placed on a table and dissected as can a brain. Nor, it seems, can mind be placed on a philosophical table and defined and described [287, p. 104].

Sociologists have indeed tried to define mind, or rather consciousness, but their accounts have still remained within the boundaries of social construction *per se*; no attempt has been made to redefine consciousness and rationality

in a more universal manner. In [298], for example, an attempt is made to redefine consciousness cross-culturally, but in the end of the article, it all falls back to a consciousness determined by social-cultural discursive definitions, ignoring, thus, its neurological basis. During an interview with a Soviet mathematician, the interviewee mentioned that shortly before sleep he could imagine certain topological spaces and thus proceed to a proof of a theorem without any need to write down notes. His specialisation was in general topology, the main objective of which is to generalise the common perception of space, based on a generic perception of spatial nearness. This personal account, when considered under the social constructivist approach, would be unaccountable, or simply ignored. But when considered under the proposed framework, which takes into account the neurological basis of consciousness, it leads to a concept of science as an *altered state of consciousness*, and as an alternative to psychoactive drugs.²⁷ Moreover, it demonstrates the tremendous effect of material culture on the human self. Besides, any inventor, and any scientist, during her trial and error method, has to *imagine and plan* the course of trial and error, just like a Neanderthal had to *imagine and plan* the complex construction of a sophisticated Mousterian tool; and this *imagining and planning* are the *psychoactive effects* of material agency.

Imagination takes place, as an experience, within the framework of fantasy. In order to explore the concept of fantasy, we have to resort, once again, to biological evolution, and its emotional and psychological effects on human evolution.²⁸ It is a common scene, when a mother cat walks, her kittens always follow her; wherever she goes they are always around, not very far away. The same happens with a female dog with her puppies and so on. This is a lay person's observation of what came to be called *attachment theory*, a cybernetic theory of human emotional bonding [62]. It had already been observed quite early in the 20th century that children have fantasies, as well, without necessarily resorting to neurotic stress [246]. Part of the children's

²⁷Another well known example is Kekulé's dream and his discovery of the aromatic chemical bond (see [396])

²⁸In older psychological-psychoanalytical literature the spelling *phantasy* used to be reserved for technical description of the workings of the unconscious.

fantasies are *transitional objects*, that is, certain kinds of artefacts used by children to alleviate their anxiety when the parent, for example, is away and they feel alone, or just before they sleep [461, 353]. The most well-known such example is a teddy bear; it could also be a blanket that the child feels secure with. Although the lay view is that children attached to certain inanimate objects are, or seem, insecure, infact this is an evolutionary adaptive mechanism, common to all mammals and birds [3, p.84-89], [154].²⁹ The evolutionary function of this mechanism is to keep the young and still weak members of the family in constant alert as to the distance of parental figures in order to increase their survival probabilities. Additionally, the parents themselves are on alert, as well, as to the potential loss of their young. The semantics, therefore, of this systemic unit is *distance* between family members. Attachment theory, in other words, “is essentially a regulatory theory, and [emotional] attachment can be defined as the interactive regulation of biological synchronicity between organisms” [403, p. 23]. Fantasy, in the context of attachment theory, becomes an *internal working model*, that is, “a representational system that allows us, for example, to imagine interactions and conversations with others, based on our previous experiences with them [66, p.103]. Fantasy, therefore, is a systemic mediating mechanism of emotional regulation in animal family groups, birds and mammals in particular.

Following up on the previous sections of this chapter, we have to consider that any later development in the cognitive functions of the later species of the *Homo* genus, took place by embedding itself in the ability to fantasize: *fantasizing predates stone tool use*. Since fantasy takes place unconsciously, and stone tool use is a conscious activity, research recently has been directed to the construct of *working memory*, which “serves to focus attention by maintaining memory representations (plans of action, short- or long-term goals, or task-relevant stimuli) in a conscious state despite interference or response competition” [201, p.S150]. The Neanderthals, for example, now are generally considered to have acquired a high level of expertise in stone tool construction, though the capacity of their working memory was limited, as documented by the rarity of stone tool innovation [468]. Any proof in math-

²⁹In the context of bird attachment ethologists talk about *imprinting*.

ematics demands a highly developed working memory mechanism, especially a long one written over many pages; one needs to bear in mind that a proof has always to be read, in order to be understood, or, in the case of innovation, many facts leading to a potential proof have to be remembered and used as leads. Activity, in other words, is *modularly organised in operational units*. Neanderthal everyday activity was organised in a similar way (see Fig.2.12, both figure and text taken from [201, p.S160]). But the stone tools in the case of a mathematical proof are no longer material; they have instead phantasmatic support, though the use of the hands has not yet been eliminated, but refined, with the use, in a similar way, of a chalk and a blackboard, or a pencil and some paper. The main components of working memory that have been so far identified are four: the *phonological store*, which is related to sound language; the *visuospatial sketchpad*, which holds visual and spatial information for a short period of time; the *central executive*, which has “overall attentional control of the working memory system” [214, p. 21]; and last, but not least, the *episodic buffer*, which brings together various sources of information into the working memory brain system (see [214, p. 1-36]; [24] is currently the most authoritative source on the working memory model). One could very well argue that perfumes and food dishes are also artefacts, mainly ignored by cognitive psychology, and it cannot be claimed that working memory is not necessary to a perfume maker, to a professional chef or to a connoisseur. There have been expressed some complains as to the potentially misleading construct of working memory [338]. In a similar fashion, it has been pointed out that different theorists have proposed different versions of working memory, and these have been invoked to explain virtually all cognitive psychology research projects [373]. Despite these criticisms, the concept of working memory is still useful for the claims of this thesis.

Although working memory, or working attention, directs attention to a task, or to an object, it still does not, as a concept, answer to a very crucial question: to where *exactly* is attention directed at? During many of the interviews conducted, many interviewees, when asked specifically to explain and elaborate more on their past or present research, they used hand movements, i.e. gestures, as part of their description narratives. That use of

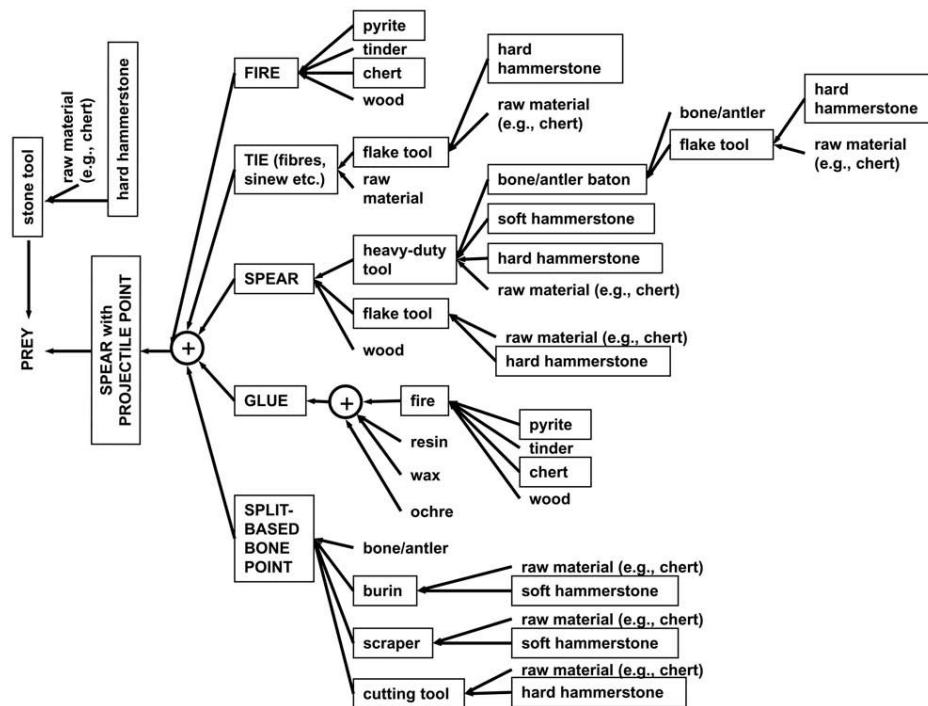


Figure 2.12: Effective chain of raw materials and tools that can be used to hunt with a spear with a split-based bone point. Boxes indicate small operational units that can be put together in a modular way [201, p.S160].

gestures, was very similar to the situation when one uses gestures to describe to somebody else a house or a building. Herein landed into the scene the concept of *intentionality*, “a generic term for the pointing-beyond-itself proper to consciousness” [163, p.109]. Simply put, intentionality is the human mind’s ability to think *about* something, whether that something is an object, a task, a concept, one’s social image, or other people’s thoughts; intentionality is always *about* something. In the case of tool construction this *aboutness* is about a future form of the stone, it is *foresight*. The form of intentionality that concerns this thesis is similar to the pointing of infants. It is a well established fact that human “[i]nfants begin to point to things for other persons from around 11 to 12 months of age” [442, p.705] (see also [83]; and for animal pointing see [233, 274].). Two things are interesting here: infant pointing starts before language acquisition; and pointing is actually a nonconventional, symbolic gesture. Both mean that actually pointing is a

more general trait of *Homo sapiens*, which goes beyond cultures. Another way of pointing during infancy is following the mother's gaze. It is the same as when many people are looking up into the sky, and when we suddenly see them we feel an urge to look up, as well. The function of pointing is taken up later in childhood by language use, with the employment of *speech acts*, that is, language phrases which mobilise into social action.³⁰ The situation in mathematics is very similar, with one exception: the mathematical formulas embedded in human speech are the artefactual fingers pointing to mathematical invisible objects. The phrase "Let there be an n-dimensional Brownian motion $\{B_t|t \geq 0\}$," points actually to a mathematical object, but those without a background in probability theory or mathematical finance, will be pointed to nothing. This pointing, although a human universal, has become in mathematics symbolic and institutionalised. A mathematical proof is actually an *intentional pointing*: it points to the reality of mathematical objects, and under this light mathematics will be studied in this thesis.

This mixed social world of an infant's attention, a world of real and imaginary objects, will later be socialised and organised by the parents' incessant use of language. And this reorganisation of cognitive reality persists for a lifetime. It must always be born in mind that the main "function of the human nervous system at the level of the cerebral cortex is the construction of a vast network of [neural-cognitive] models of the self and the world" [269, p.365]; the totality of these neural-cognitive models comprises the *cognized environment* of an individual, in contrast with his or her *operational environment* "which is the real nature of both that individual as an organism and that individual's world as an ecosystem" [*Ibid.*] (see also [378, p.22] for the origin of these terms). In other words, genetically determined massive systems of neural structures produce human knowledge, and these systems have been shaped, as we have seen so far, by artefact construction and use. The cognized environment, which produces the human experience, is heavily based on *neurognosis*, that is, on a genetically-neurally preorganised epistemic mind, which, in the course of its life reorganises itself continuously

³⁰In fact some authors refer to infant, and possible animal, pointing as *pre-speech acts* [233].

through creating countless new neural synapses (see also [271], and [270] for the similar concept of Jungian archetypes). Knowledge, therefore, can be considered as a *materially induced altered state of consciousness*. Even telling a funny joke among a group of people, and then have everybody laughing involves hypnotic states [81]. In fact “many pre-industrial societies are ‘polyphasic’ ... that is, they value altered states [of consciousness] as sacred and socially constructive, whereas the post-industrial West is ‘monophasic,’ that is, it exclusively valorizes a waking state that is assumed to be predominantly ‘rational’,” [457, p. 181] (see also [397, 458], also [384] describes the quite interesting concept of a *cultural neurohermeneutic system*). It has been argued, moreover, that *behavioural modernity*, that is, the appearance of cave art as the main characteristic of *Homo sapiens* differentiating him as a species from the Neanderthals, was actually the *widening of the spectrum of consciousness*, which lead the path beyond tool construction: cave paintings, figurative and decorative art, and full-blown language actually have no apparent practical use, except for social-aesthetic purposes [287, 101-136]. Considering the fact that actually stone tools and human consciousness are more intimately connected than generally thought, it can be, and will be, additionally argued that actually the new tool abilities of *Homo sapiens* are the abilities of producing material narratives, and the ability to produce, in particular, sophisticated imaginary material mathematics; and it will become evident how narrative, actually, unifies human consciousness, language and materiality, and mathematical proof is just one aspect of that.

2.5 The Rise of the Social Constructions

Very recently a scientific article was published that presents

data [that] support the hypothesis that a gene-culture co-evolutionary dynamic between tool use and social transmission was on-going in human evolution, starting at least 2.5 mya [million years ago] and potentially continuing to the present [330, p. 2].

The organisation and structure of the underlying ingenious experiments, though, were as important as its results from an STS point of view. A sample of 184 adults was assembled and each participant would be tested on his or her performance on stone knapping, and then he or she would be tutoring on stone knapping the next participant who, in turn, would be tested on his or her stone knapping performance, and so on. The participants were divided into five different transmission chains according to the way of being taught stone knapping (see Fig.2.13):

- (i) *Reverse Engineering*: pupils were provided with a core and hammerstone for practice, but saw only the flakes manufactured by their tutor and not their tutor themselves;
- (ii) *Imitation/Emulation*: in addition to having their own core and hammerstone, pupils also observed their tutor making flakes, but could not interact with them;
- (iii) *Basic Teaching*: in addition to demonstrating tool production, tutors could also manually shape the pupil's grasp of their hammer stone or core, slow their own actions and reorient themselves to allow the pupil a clear view;
- (iv) *Gestural Teaching*: tutors and pupils could also interact using any gestures, but no vocalizations;
- (v) *Verbal Teaching*: tutors and pupils were also permitted to speak. [330, p. 7, my italics]

The first link in each transmission chain was a trained experimenter.

The idea behind this experiment was to test the fidelity of information transmission over successive teachings along each chain. Each teaching method, including self-teaching (i.e. reverse engineering), was conducted over six chains, four small, and two longer, chains, and the researchers had developed six measures of performance. A major difference between human cumulative culture and animal cultures is the ability of humans to *transmit information with high fidelity*, and teaching quite possibly has “evolved in humans because cumulative culture renders otherwise difficult-to-acquire valuable information available to teach” [152, p. 2760] (see also [85, 115, 285]). Given

that stone tool knapping is cognitively very demanding [395], and taking into consideration that early hominids had to survive in a very demanding ecological environment [369], teaching would speed up the process of acquiring the necessary cognitive skills to provide “access to the expanded array of foods made available by sharp and effective crushing implements” [369, p. 10] such as the biface Acheulean hand-axes. The major objective of this experiment, in other words, was to statistically test the co-evolutionary relationship between stone tool making and variations of a particular mode of social construction, that is, teaching.

The experimenters found that stone tool construction performance was greatly enhanced in almost all ways of teaching, with two exceptions: performance after reverse engineering and after imitation/emulation dropped to the level of the individual’s intuition and personal cognitive abilities. Verbal teaching, on the other hand, provided, in general, the highest fidelity of information transmission. Even gestural teaching enhanced personal performance and maintained it along each chain; still performance after gestural teaching was rather low, when compared to verbal teaching. The authors of the article suggest that gestural teaching probably corresponded to a proto-language which maintained construction performance over generations of early hominins to a level satisfactory for species survival and, moreover,

the appearance of [biface] Acheulean tools 1.7 mya [million years ago] reflects, in part, the evolution of mechanisms of transmission that facilitated the more effective transmission of Oldowan tool-making, but also enabled the reliable transmission of the sub-goals and techniques required to make the distinctive and regularly shaped Acheulean tools [330, p. 6].

Gestural teaching, in other words, might have been the first appearance of social construction among hominids which, as the authors of the article suggest, was a contributory factor to the evolution of the *Homo* cognition. Moreover, gestural teaching, demonstrates, in the opinion of the author of this thesis, the importance of materiality in social constructions: every social construction happens through a particular *material spectacle*, and this

material spectacle is what *induces social constructions*.³¹

Still though, to explain the appearance of modern mathematics, such as the advent of infinite-dimensional Hilbert spaces in today's quantum physics, stone tool construction combined with gestural teaching is not enough. Full verbal teaching is not enough, as well: Einstein's theory of relativity requires four dimensions of spacetime, and there are other gravitational theories, such as superstring theories, which use mathematical geometrical models of ten dimensions. Ten-dimensional artefacts, though, do not exist, and even if they do exist, if we accept modern physics to be valid in its conclusions, they remain in a realm not accessible to usual and everyday human cognition. Mathematicians, on the other hand, have no problem in mentally handling them when writing down a (two-dimensional) proof with this kind of ten-dimensional, or even infinite-dimensional, artefacts. And that is, exactly, the point where mathematics meets art, religion and shamanism: visual access to these multidimensional transcendental space entities is provided by the neurologically duplicate brain of a shaman cave artist who can "see" the "spirit worlds" of spacetime, superstrings, or noncommutative geometry. And the interesting thing about shamanism is that astrology today, though generally discarded as a science by the modern scientific community, is not only alive, but very well kicking with both of its legs: everyone, whether scientist or not, knows his or her star sign; if two star signs can pair well, according to astrological wisdom, then their marriage will be very blissful.

The usual answer given when one is very good at mathematics is that he is clever, that is, mathematics is a matter of exceptional specialised intelligence. And indeed human, as well as animal, intelligence has been claimed to be modular (see for example [164, 244]). The "great transition" from Neandertals to modern humans, then, was that these initially independent brain modules began to communicate with each other (see [325]). A major opposing position to this view of modular intelligence as the all-resolving and all-encompassing concept that explains "everything in human cognition" has been David Lewis-Williams's view:

³¹More on this in the next chapter.

[T]he emphasis on intelligence has marginalized the importance of the full range of human consciousness in human behaviour. Art and the ability to comprehend it are more dependent on kinds of mental imagery and the ability to manipulate mental images than on intelligence [287, p. 111].

Under this light, human intelligence is a Western construction and intelligent people are those “becoming more and more like Western scientists” [287, p. 111]. Modern human consciousness, in Lewis-Williams’s view, became *internally more expanded* than Neanderthal consciousness: evolution, in other words expanded the *spectrum of consciousness*, that is, the continuum from full cognitive alert, to a state of being half-awake, half-asleep, to dreaming and deep unconscious sleep. The spectrum of consciousness, in other words, is a continuum between two states: one of *alert [i.e. purely extrospective] consciousness* and one of *autistic [i.e. deeply introspective] consciousness* (see Fig. 2.14), and the centre of scientific attention now moves to “the neurology and functioning of the brain [that] creates a *mercurial* type of human consciousness that is universal” [288, p. 9, italics added]. And a mercurial consciousness means one thing: instead of talking about *one* (type of stable and crystallized) *consciousness*, we should be talking about *altered states of consciousness* moving in a continuum between alert, that is, *purely extrospective*, and autistic, that is, *deeply introspective*, states of consciousness. Lewis-Williams, though, quite wisely hastens to comment that

[t]he phrase “altered states of consciousness” is useful enough, but we need to remember that it carries a lot of cultural baggage [287, p. 126].

The theory that religion and its material expressions such as cave art are hardwired on the brain had already been formulated earlier (for example [86]), and later on became more well known in the archaeological community, as one branch of *post-processual*, or sometimes *interpretative*, archaeology.³² Lewis-Williams, in particular, proposed that cave paintings can be explained

³²For a good overview of archaeological theories from past to now see [230].

as the products of prehistoric shamans: instead of passing into “naturally induced” autistic states of consciousness, prehistoric shamans artificially induced states of consciousness through psychoactive chemicals found in particular plants, or through repetitive sounds combined with ritual dance and singing: they moved along an *intensified* spectrum of consciousness in three stages leading up to hallucinations (see Fig. 2.14). Lewis-Williams’s three stage model has received considerable criticism, one strand of which is that it “inherently seeks to understand the meaning of rock art based modern or historical San narratives” [315, p. 226].³³ Moreover, while inducing trance through the consumption psychoactive plant substances is indeed possible, it seems rather implausible since “in Europe, especially where caves containing Upper Palaeolithic cave art are located, the evidence for hallucinogenic substances is very sparse” [213, p. 91] (see also [98]; for more recent debates see [160, 212] and the references therein). In spite of the criticisms over the particular causes, though, the idea of a hardwired expanded spectrum of *Homo sapiens* consciousness, and the corresponding altered states of consciousness have remained resistant to criticism, and, in the author’s opinion, provide the missing link between human biology, human consciousness, and human societies.

The idea of the spectrum of human consciousness as a continuum between alert and autism, in the author’s opinion, goes far beyond Lewis-Williams’s original intentions to describe the neurobiological origins of cave art, and later on early religion. Alert, in Lewis-Williams’s definition is the state of “waking”, and “problem-oriented thought” (see Fig. 2.14). But what exactly is waking, and what exactly is problem-oriented thought? If we consider a mathematician *deeply absorbed in* a solution of a difficult problem, then he or she is obviously awake, in a state of intense problem-oriented thought, but also in a deeply autistic state of consciousness. Or someone who is talking on the mobile phone with a creditor using hands-free equipment, while crossing a street and at the same time checking both sides of the street to *alert* himself or herself of any dangerously approaching vehicle, is indeed in a state

³³The San people, formerly also known as Bushmen, are indigenous hunter-gatherers in the Kalahari desert of the South part of Africa.

of alert, but at the same time he is or she is in an autistic state of being *absorbed in* the conversation over negotiating a better financial deal. Moreover, when a university student is cramming for a whole week to prepare for the incoming exams, we could very well say that he or she is temporarily in an intensified trajectory of states of consciousness (see Fig. 2.14), since cramming means both overloading short-term memory with lots of information and, quite often, extended sleep deprivation.

Altered states of consciousness, in other words, are part and parcel of the “normal” everyday life of all the members of any society, either Western or not, and is, therefore a universal phenomenon. When mathematicians, in other words, are reading and writing the proof of a theorem in infinite-dimensional Hilbert spaces they are in an altered state of consciousness; a lay person can see only the letters and the symbols of the text of the proof, but cannot “see” the mathematical (fictional) entities employed in the proof. All the mathematicians interviewed for this thesis mentioned extended and long periods of studying, especially while being undergraduate and postgraduate students. And studying in mathematics means long hours, isolated from the community, and solving problems; a disciplinary regime quite similar to a monk’s, but instead of solving mathematical problems, a monk prays *in communion* with the christian God. But both a diligent mathematician and a pious monk have reached their particular states of consciousness without resorting to ritual dance or psychoactive substances as a shaman would have: they have *domesticated* their altered states of consciousness through extensive and persistent practice.

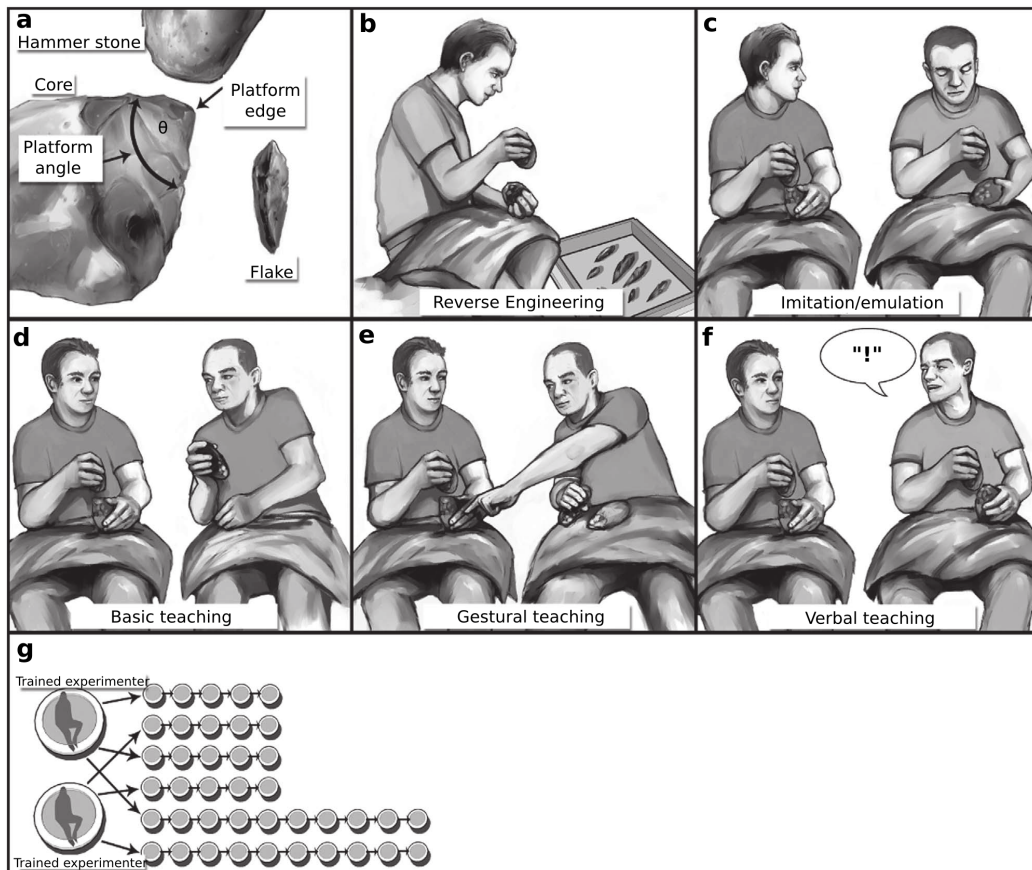


Figure 2.13: *Experimental design and structure.* (a) A diagram of the stone knapping process. The hammerstone strikes the core with the goal of producing a flake. The platform edge and angle are important to the success of knapping. (b–f) The five learning conditions. (g) The structure of the experiment. For each condition, six chains were carried out (four short and two long); one of two trained experimenters started each chain (equally within each condition). (Both figure and text taken from [330, p. 2]).

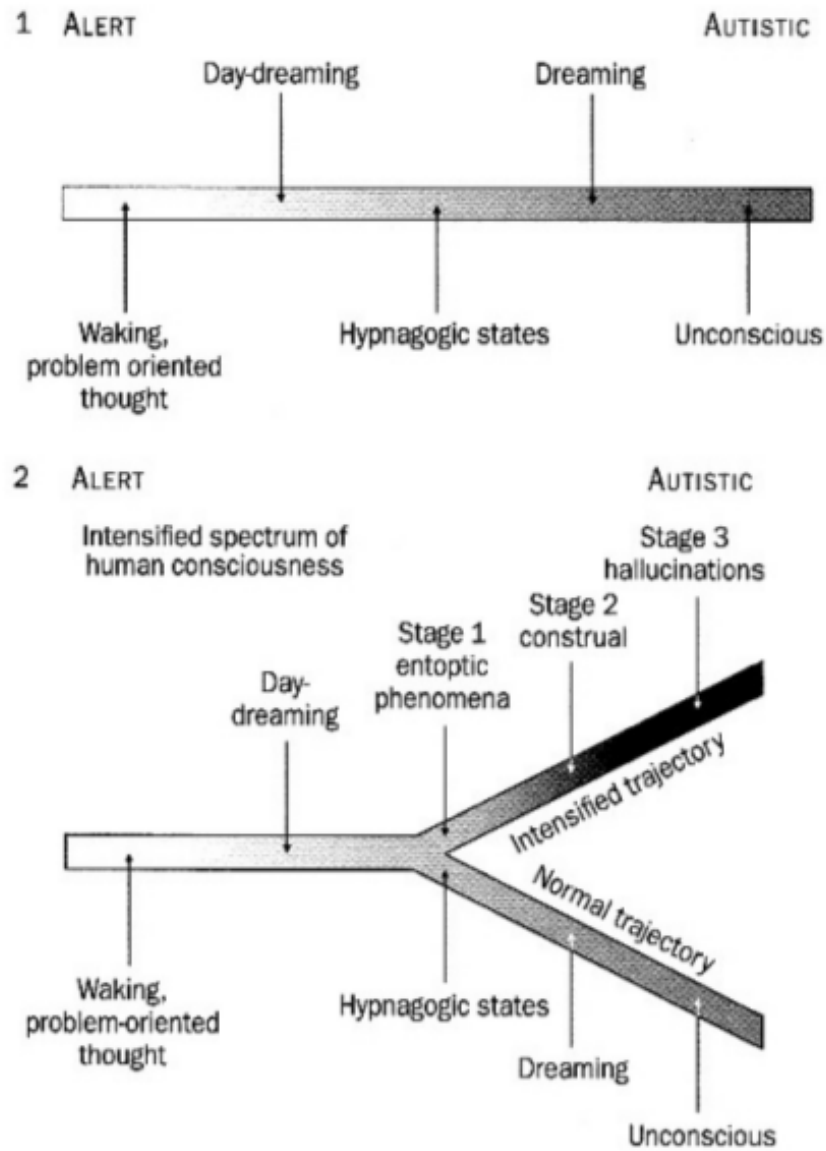


Figure 2.14: The two spectra of consciousness: (1) “normal consciousness” that drifts from alert to somnolent states, and (2) the “intensified trajectory” that leads to hallucinations [287, p. 125].

Chapter 3

When the Mind Talks to the Hand

3.1 Introduction

In the summer of 2011 I was for a month in Moscow in a preliminary engagement with potential sources of data, mainly archival. The first member of staff employed in the Lomonosov University (MGU) I came across was a mathematical physicist around 40 years old, who was running a laboratory on computational weather models. That was not his specialty; he had specialised in quantum physics, the so-called “string theory,” in particular, a very arcane mathematical subject. We arranged a meeting at the park in front of the Main Building, and I was keen on meeting him, thinking about what kind of questions to ask him about the MGU. ... I met with Aleksei¹ at around twelve o'clock at noon, under the discreet gaze of Lomonosov's statue. The sky was clear of any clouds, and the sun was shining with a nice complacent grin. We greeted each other, exchanged the customary ceremonial courtesy of enquiring on each other's health and general state of mind, and we began walking towards a direction he seemed to be familiar with. We went into the building of physics, located on the left front side of the Main Building, he showed me his laboratory, along with the small-scale supercomputer main-frame he was using to run his weather models, and then we decided to go for a walk to Sparrow Hills, a very well known and popular natural park along the Moskova river, fifteen to twenty minutes' walk from the main campus.

After a rather interesting walk filled with discussion and argumentation of academic nature we reach the Sparrow Hills park and begin walking down towards the river bank. We get to the bank where there are lots of people strolling up and down: a mother pushing a baby cart ahead and talking to her hands-free mobile, a grandpa with his two grandchildren listening to his stories, a few young couples walking hand in hand and occasionally kissing each other, some youths, probably university students, lying on the grass, talking loudly and arguing over some subject, football teams I think. We buy beers from a street vendor, walk a few meters, find some free place and sit down on the grass. As we discuss, at some point the topic of the conversation gets to string theory and while Aleksei is trying to formulate

¹This is not his real name.

an idea, or rather acquire a better view of what he already knows, of a superstring in a ten-dimensional (geometrical) space, he uses his hands, as if he really held a string between his extended fingers. He then moves slowly each hand away from the other, while grasping with his extended fingers, thumb, index, and middle, a “real” and “material” string, which is sliding, at the same time, between his grasping fingers (see Fig.). During his thinking aloud I am closely following his train of thought, while at the same time I can “look at” this “concrete” superstring sliding through his grasping extended fingers, as he moves away slowly one hand from the other.

This seemingly unimportant incident, which actually prompted the previous chapter, leads to some very interesting observations-questions. It seems that conversations, and in fact *every single* conversation, to be related to the flow of time, or rather, to be a cinematic movie, with the interlocutors “watching” the unfolding scenes, while at the same time each one and all of these participators “direct” their own scenes. The hand movements, along with the voice intonations play a major role in that. Assuming that any conversation *is indeed* a cinematic film, then who are the protagonists, who are the screenwriters, and who are the “material” film producers? If there are discussions on music or on dance, then what exactly the spectators-directors “see”, and especially “watch”? Focusing our attention to mathematical discussions, lectures, or workshops a certain very important and pressing question draws itself forward: how are Hilbert spaces, that is, infinite-dimensional modern versions of the ancient Greek Euclidean geometrical spaces, being conceived and communicated through proofs? How is it possible to conceive these kind of infinite-dimensional objects using only up to three-dimensional artefacts, such as paper sheets, black boards, or solid maquettes? In the “superstring incident,” the superstring materially, simply, did not exist, but the mathematician *describing* it, along with the listener *watching* it, would argue to the contrary. The author would actually redescribe the whole incident as a case of an *everyday-practised shamanism*: the superstring along with its ten-dimensional space was a shamanistic vision, which, at the same time was certainly not an instance of psychotic hallucination, which would persist over time. Or was it?

3.2 The Movement of the Mind

Southern Italy, besides its traditional cuisine, is very famous for the hand gestures its people use in everyday conversations. Hollywood has repeatedly used this fact in many films depicting the deeds and misdeeds of organised crime figures of Italian descent, both real and fictional ones. In Fig.3.1 we see three scenes with Robert De Niro, a well known actor who has repeatedly impersonated fictional American crime figures of Italian descent. De Niro himself is partly of Italian descent, and grew up in Brooklyn. As part of his performances, in order to perform as realistically as possible, he has been combining in a very consistent and flowing manner Brooklyn American English accent, which is usually associated with New Yorkers of lower social status,² along with a lot of Italian-descended gestures. But besides the potential ethnic origins of a gesture, some things seem to be universal. On the left screen shot, although we cannot listen to the dialogue, from the hand gestures we can deduce that probably De Niro is talking about something *related to him*, such as “Why are you doing this to me?”, or “I did so much for you, and you double-cross me?”, and so on. On the upper right picture, we can clearly see that De Niro addresses his words *to another* speaker, such as “You are a very clever fella!”, or “Careful, you’re walking on a fine line here!”, and so on. But what is interesting for this thesis is the third screen shot, which quite eloquently summarises the argument of this chapter.

On the lower right screen shot we see De Niro with Ray Liotta³ sitting opposite at a table. Their hands are leaning against the table and De Niro holds from a corner a pack of paper sheets with the fingers of his left hand. De Niro seems to be talking and Liotta seems to be listening. From a first look, although these postures and this momentary glimpse of hand gestures seem quite usual in everyday life, there is actually one point asking for further clarification. From a biological point of view, both of their bodies are resting, partly on the chairs, through sitting, and partly on the table through their

²See the groundbreaking [263] for the various New York City accents and their relation to social status.

³This screen shot is from *GoodFellas*, directed by Martin Scorsese.



Figure 3.1: Robert De Niro: scenes from various films. His hand gestures can be clearly seen.

lower arms. It does not, therefore, make any sense, from a biological point of view, that De Niro is holding out a pack of papers with one of his hands, since this complicates his body balancing; he has to exert extra effort, that is more energy, to keep his upper body in an upright position. This contradiction can be easily solved if we assume that this pack of papers is actually part of De Niro's statement: either, for example, the pack itself is important, or using the pack's size to emphasize his point, or maybe both. So what we could conclude from this screen shot is that the conversation can be actually about something not visible to us, and since the body expends more energy than it would be required normally from a simple sitting position, means that actually gestures, on the face of it, are actually more important than generally thought, and actually gestures *can* involve artefacts as well.

If instead of a paper pack, De Niro was holding out a chalk, like the one used in mathematics classrooms, he could have used it in a similar way, to point to something, or to emphasize something he says. And the importance of an artefact, at least a portable one such as a piece of chalk, or a mobile telephone, for example, is that it can be used as pointer to something, or rather in a (linguistically) pragmatic way, as an extension of a hand gesture. So the problem which is going to concern us in this chapter is the use of artefacts and mathematical symbols as utterances, rather than sentences. The main

“difference between sentences and utterances is that sentences are abstract, not tied to contexts, whereas utterances are identified by their contexts” ([192, p. 6]) And a mathematical proof, no matter how abstract it seems to be, is actually anchored on social context: written language along with special symbols employed by mathematicians actually record the social objects created during the course of a discussion between mathematicians, or during instruction in the classroom. And the recording artefacts, such as written language on a published paper, or on the blackboard, have the same effect as sound language and hand gestures on human working consciousness, that is the consciousness at the time of a social interaction. The fact that this holds for both high school Euclidean geometry, which is easily accessible to a lay person, and for modern infinite-dimensional geometry, which is accessible only to professional working mathematicians, only enhances the argument: there are many self-taught mathematicians in ancient Greek Euclidean geometry, but almost none in infinite-dimensional geometrical spaces, the bread and butter of modern quantum physics. Class instruction, a purely social event, becomes extremely important in the understanding of very advanced mathematics. Even child prodigies⁴ have to be introduced at some point to mathematics, and this introduction is always going to be some kind of social event.

Hand gestures are actually not restricted to humans. There is a lot of ongoing research on gestures of nonhuman primates (see Fig.3.2). Extending gesture research to nonhuman primates, however, poses some new challenges as to how can a gesture be distinguished from instrumental actions such as throwing or pushing an object. Primate ethologists, for example, have considered *hand clapping* in chimpanzees [441, p. 141], or *chest beating* in gorillas [358, p. 100] as gestures (see [78] for a more detailed view of recent research on nonhuman primates). From a more practical point of view a primate gesture could be classified as such as to “whether (i) it is motorically ineffective, (ii) there is response waiting, (iii) gaze alternation, and (iv) persistence” [290, p. 119]. There are however differences between humans and nonhuman pri-

⁴Probably one of the most well known “celebrity” child prodigies in mathematics today is the Australian mathematician Terence Tao (see also [193]).

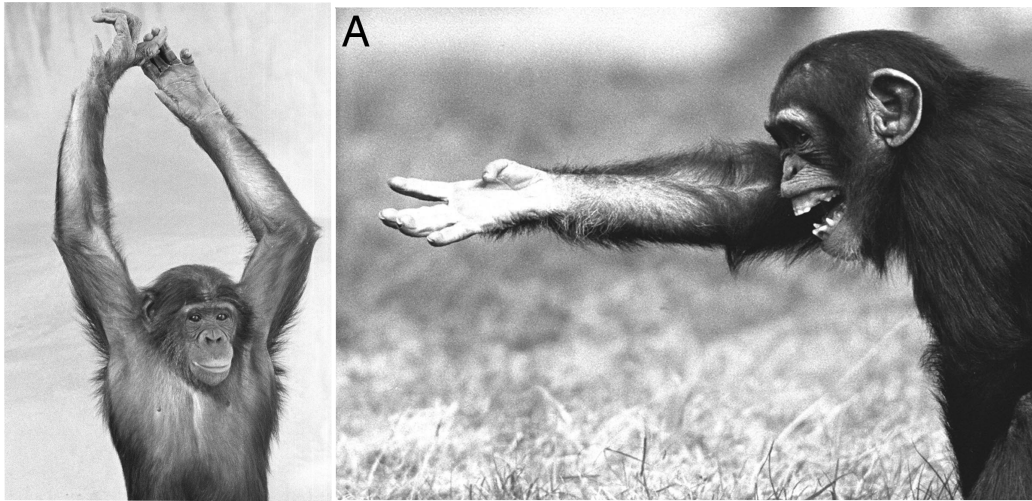


Figure 3.2: **Left:** An adolescent bonobo male making sexual advances to a female adds the *arm raise* gesture. **Right:** A juvenile chimpanzee tries to reclaim food that a dominant has taken away by combining the *reach out up* begging gesture with a *scream* vocalization .[365, p. 8185, emphasis in the original]

mates in their gesturing. Human infants, for example, are generally known to point in order to share attention with an adult to an object, to show towards the direction of an object, or to instruct an adult to do something for them [292]. Apes, on the other hand, while they do deploy *imperative* pointing gestures [76], that is, gestures requesting for something such as food, they do not engage in *declarative* gesturing mode, that is, gestures leading to shared attention [77] (see also [440]). Despite the various differences among humans and nonhuman primates, there are actually many commonalities in gesturing. These commonalities have led to the suggestion that gestures predate speech from an evolutionary perspective [172, 357], a suggestion which seems to be supported by both neurobiological evidence and the mirror-neuron hypothesis [106, 448]. In humans, in particular, there is a common neural substratum between gestures and speech [30, 417, 470]. There is, on the other hand, the view that language and gestures co-evolved, and none predates the other and it is based on a mirror neuron ‘twist’, the so-called *Mead’s loop*: “[t]his ‘twist’ is what we hypothesize evolved – a new kind of response, a self-response to one’s own gestures via mirror neurons” [320, p. 66].

No matter what the origin of language may be, whether gestures predate speech or the reverse, or whether tool construction predates language or not, as examined in the first chapter, one important fact remains unchanged: speech, gestures and materiality are based on a common neural substratum. Their relationship is so deeply ingrained into human cognition, that ignoring this fact *can undermine the social, and indeed the whole of scientific enterprise*. There is no science without artefacts, whether these be laboratory, scientific article, or even discussions. Human voice is, in fact, a totally neglected artefact, whose material, like the material of any artefact, is something prevalent: the surrounding air. Just because we are very well equipped biologically to handle draughts of air, by producing artificial wind called breath and voice, does not necessarily mean that we are not affected by it. Our whole respiratory system *is our third arm*, and our mouth is *our third hand*. We cannot simply ignore this fact in the social sciences just because it is not a current research trend, and just because the university departments of the social sciences have not included in the curriculum other disciplines such as biology, archaeology, or cognitive science. Materiality is enough actually to undermine the illusion of direct observation: what exactly is being observed in the social sciences? The ethnographic notes of a field researcher are *second-order observations*, that is, observations of observations, not first-order observations; class instruction is second-order observation, not direct observation; even a scientific article is based on second-order observation by simply citing other past articles. Philosophical treatises and religious experience, the most abstract or transcendental of experiences are second-order observations: books along with instruction are being observed in order to reach the ‘next’ level of knowledge or experience. The first step to penetrate this materiality-observation circle is to start first with hand gestures, that is, *the organic materiality of knowledge*.

Gesture studies, actually, can be traced back to Roman antiquity, when the Roman rhetorician Quintilian wrote his treatise on oratory ([372]; see also [133]). Since then many authors over many periods of Western history have dealt with gesture as a subject of a treatise (see [239, p. 17–83]). The most important development in gesture studies was, in fact, outside of the so-

cial sciences: the inventions of photography and cinema. With photography, and later with cinematic technology, researchers were in position to collect *material data*, that is, data ignoring the ravages of time, available to the subsequent generations of researchers by means of proxy observation, and most importantly: materiality rendered data a sense of objectivity. One of the pioneers in post-war gesture studies was David Efron (see [138]). Although no film data have been found, Efron’s “important analyses of the different ways in which gesture can be employed with speech owe much to the fact that he was able to employ film” [240, p. 19]. Gregory Bateson also was one of the pioneers in photographic data collection for the social sciences in his field work in New Guinea [33]. Bateson, actually, later commented that “human kinesic communication, facial expression and vocal intonation far exceed anything that any animal is known to produce” [34, p. 614–615]. A major influence in gesture studies, a rather unexpected one, was Noam Chomsky’s *language acquisition device*, a conjectured brain module which helps a child develop and internally represent “a generative grammar [...] on the basis of [personal] observation of what we may call primary linguistic data” [91, p. 25], that is, on the basis of the child’s personal linguistic experience and interaction with his or her social milieu. Chomsky proposed, in other words, that language acquisition is, in fact, a biologically predisposed human ability. Although he has gradually abandoned this particular theory in favour of a *Universal Grammar* (see, for example, [402, esp. p. 1–30]), the idea behind Chomsky’s proposal still remained the same: children are born with the ability to learn language.

Gestures were actually first studied systematically in connection with phonology [236]. A *segment* in linguistic phonology is “any discrete unit that can be identified, either physically or auditorily, in the stream of speech” [108, p. 426]. The word “laugh”, for example, as we can see from its phonetic transcription in Oxford’s Advanced Learner’s Dictionary, that is [la:f], consists of three sound segments, while the word “leisure” consists of four segments, in the case of standard American English, i.e. [l'i:ʒər], and three to four segments in the case of standard British English, i.e. [l'eʒə(r)], depending on whether a vowel or consonant or no sound follows. It is rather obvious

that there is not always a one-to-one correspondence between the written spelling of a word and its corresponding sound segments in actual speech. Sound segments are, in fact, more important for gesture studies, since they are among the first material modalities of language that a child learns, along with hand gesturing. Analysing the fundamental sound segments is still not enough. The word “leisure,” for example, consists of two *syllables*, as we can see from its phonetic transcription, either its American or its British version. The word “Aberdeen,” as we can see from its (standard British) phonetic transcription [Abəd'i:n] consists of three syllables, but of two *feet*, that is, of two “phonological constituent[s] above the syllable and below the word” [196, p. 214]; the two feet of the word “Aberdeen” are [Abə] and [d'i:n], “the middle syllable [of which] is very much shorter than either the first or the third, both of which occur in the strong position of a foot” [196, p. 215]. A foot is distinct, in general, by its two syllables: one strong and one weak, and its name derives from metric poetry. In fact there is a *phonological hierarchy* involved in speech utterances (see Table 3.1).

It is important to analyse the phonological hierarchy of speech along with that of hand gestures, in order to proceed to a more detailed analysis of gestures. The basis of phonological analysis used in gesture studies is the *tone unit*, which corresponds to the *phonological phrase* in the previous diagram, as described, for example, in [109] (see also [196, p. 246–261] and [195, p. 296–320] and the references therein for a technically more detailed analysis). Tone, as well as stress, and prosody, are the *suprasegmental*, or *non-segmental*, elements of speech, and they are more important for gesture analysis, because they organise the units of meaning during speech. The basis of a tone unit is the *nuclear syllable*, which bears the *nuclear tone*, that is, “the most prominent pitch movement in the tone-unit” [109, p. 25]. Additionally, the “boundary features [of a tone unit] may comprise a marked shift in pitch, various types of pause, and modifications to the final phonetic segments in the unit” [p. 25]. If we denote the nuclear tone with small capitals, the contours between the tone units with a vertical dash, i.e. “|”,⁵ then the sentence in the previous diagram can be written as: “mAny pupils |were slow |to respOnd”.

⁵This notation is employed from [109] for reasons of clarity.

Depending on what the speaker wants to emphasize, the intonation between them can vary:

- ↗mAny pupils |↘were slOw |to respOnd : this intonational phrase would most probably emphasize the big number of pupils;
- mAny pupils |↗were slOw |↘to respOnd : this intonational phrase would most probably emphasize the slowness of the pupils' response; and so on.

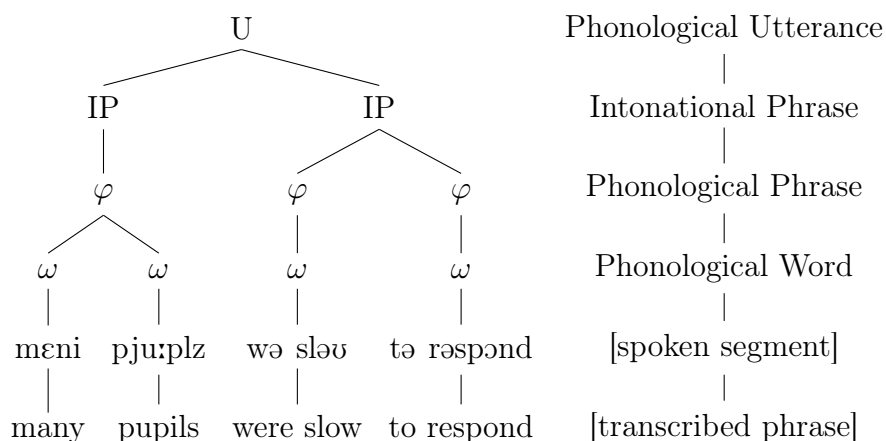


Table 3.1: Phonological Hierarchy (taken from [196, p. 246]):

It has to be emphasized, at this point that the above observations are always dependent on the speaker's intonational customs and habits, and are not universally observed. Moreover, if the speaker is of foreign descent, and his or her speaking language is not his or her native one, then the intonational organisation of his or her speech can be that of the native language, or a mixture of the native and the later acquired one and so on. It has been found, for example, that modern Rioplatense Spanish, that is, the Spanish dialect spoken in Buenos Aires, has intonational patterns very similar to Italian, or rather, to Neapolitan Italian, a fact attributed to the great number of Italian immigrants to Argentina since the 19th century [99]. In the case of this thesis, the interviewees were speaking, while gesturing, either in Russian, or in English with Russian or a mixture of Russian-North-American intonation.

Despite the different intonational customs. though, the nuclear-tone international model is the standard model accepted today by modern linguists in speech analysis across the majority of modern languages, and that is the model employed in this thesis, as well. In the case of Russian, the standard nuclear-tone model describing the intonational organisation of Russian speech is that of Bryzgunova [71, p. 772–778], which identifies seven *intonational constructions* around the tonal centre of a phrase, and is widely used in teaching Russian as a foreign language.

Corresponding to the previously mentioned phonological hierarchy, there is a gesture hierarchy (see Table 3.2). The gesture *stroke* is the most prominent

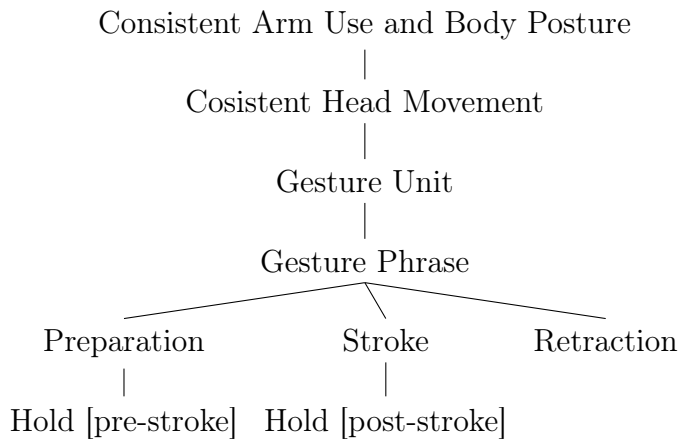


Table 3.2: Gesture Hierarchy (taken from [321, p. 209]):

and visible part of a hand gesture; that is what is commonly perceived as a gesture. Before the stroke there is the *preparation phase*, during which the hand starts moving from a resting position towards a position in front the speaker’s gesture space, and after the stroke, during the *retraction (or recovery) phase* the limb (or limbs) return to its (their) original resting position. Both the preparation and the retraction phases are not obligatory during a gesture excursion. A *gesture hold* is “any temporary cessation of movement without leaving the gesture hierarchy (in contrast to a rest, which means exiting the hierarchy)” [318, p. 83]. A *gesture phrase* is constituted by a stroke phase and optionally by the preparation and retraction phases, possibly along with various holds, which temporarily envelop each stroke. Many

gesture phrases constitute a *gesture unit*. A gesture unit is distinct from a gesture phrase in that it starts from a resting position of the limb (or limbs) and finishes with a return to a resting position. Moving up in the hierarchy, *consistent head movements* enclose the gesture units of relevant themes of meaning. Finally *consistent arm and body posture* along with simultaneous discourse correspond approximately to a “paragraph” of speech and body motion ([321, p. 210]). If we move back to speech (i.e. phonological) hierarchy, and follow Kendon’s original article [236, p. 184–190], we can see that the nuclear tone of a tone unit, i.e. of a phonological phrase, corresponds to the gesture stroke of a gesture phrase. Gesture phrases on the other hand, “coincide with and tend to be semantically coherent with the units of phasal meaning or ‘idea units’ expressed in the tone units” [239, p. 126]. A gesture unit corresponds to a (spoken) *locution*, that is, a group of tone units, which “tends to correspond to a complete [grammatical] sentence [... and] it is separated by a distinct pause from any immediately preceding locution” [236, p. 85]. Higher in the hierarchy is the *locution group*, i.e. a sequence of locutions, and at the highest level of the hierarchy is the *locution cluster* (See Table 3.3). Although phonological analysis will not be employed in the

Table 3.3: Kinesic and Phonological Hierarchies Combined (taken from [318, p. 85]).

Kinesic Hierarchy	Phonological Hierarchy
Consistent Arm Use and Body Posture	Locution Cluster
Consistent Head Movement	Locution Group
One Gesture Unit	[One] Locution
One Gesture Phrase	Tone Group [Unit]
One Stroke	Most Prominent [Nuclear] Syllable

text, it is important for the transcription process of gestures (see the *Duncan Coding Manual* in [319, p. 264–272]). Simply put, a *linguistic utterance* has one visual and one auditory component, and this “semantically coherent gesture–speech *ensemble* is a speaker *achievement*” [239, p. 127, emphasis in the original].

These early observations of gesture and speech coordination [103, 104, 236, 237] were later proposed by McNeill to be two modalities of “outer speech” which are both produced by a conjectured common process of “inner speech” [317]. McNeill has stated that his aim was to formulate a theoretical framework which “[a]t a minimum ... should explain how speech, which is linear through time, is related to the type of thinking that we see exhibited in the simultaneous gesture, thinking that is instantaneous, imagistic and global – analog rather than digital” [318, p. 11]. Since this chapter will be utilising McNeill’s conceptual framework, its elaboration, and justification, will be left for later on in the text. In a similar fashion Kita has proposed his *Information Packaging Hypothesis* to account for the coordination of speech and gesture [245]. According to Kita, two modes of thinking are involved in the coordination of speech and gesture: analytical thinking and spatio-temporal thinking. Analytic thinking organises information through conceptual templates whose activation “does not necessarily involve activation of ‘peripheral’ modules such as visual, tactile, and motoric modules, or ‘non-combinatoric’ modules such as affect” (p. 164). Spatio-motoric thinking, on the contrary, organises information “according to the features of the environment” which is “normally employed when people interact with the physical environment, using the body (e.g., the interaction with an object, locomotion, and imitating somebody else’s action).” Spatio-motoric thinking, in its turn, creates an imaginary “virtual environment” within which gestures are enacted virtual [bodily-kinesthetic] actions. The speech-gesture coordination emerges as an effort and accomplishment of the speaker. De Ruiter has extended Levelt’s linear model of information processing during the speaking process [282, p. 1–28] and has proposed a sequence of stages of speech and gesture production, during which each stage is conducted by a separate cognitive module [399]. The main extra module de Ruiter has added is the Gesture Planner, which processes gestural information, along the Formulator, which is the module responsible for speech processing. According to de Ruiter the Gesture Planner and the Formulator process information simultaneously and independently of each other.

There are, on the other hand, some researchers who have asserted that the

function of gestures is that of communicative assistance to, rather than full participation in, an utterance. Freedman and Bucci, for example, have proposed that “such [hand] such movements may aid in marking logical or grammatical relations, reducing the heavy load on short-term memory involved in generating spoken sentences” [72, p.621]. They suggest, in other words, that people are gesturing, in order to buy some milliseconds to organize their speech utterances which are to follow. Moreover gestures are employed for their parsing abilities, that is, they act as visual punctuation marks organising syntactic aspects of discourse [155, 156, 157] (see also [390]). The association between gestures and pauses has also been examined, and it has been proposed that “when retrieval of the phonological word form delays speech output, the gesture onset occurs in the silent pause preceding the word” [74, p.172] (see also [73, 200]). It has also been strongly advocated that hand gestures could facilitate lexical retrieval and corresponding cognitive models have been proposed as to that [257, 258, 259]. In one particular experiment, for instance, conducted to test the correlation between muscular activity and lexical retrieval, participants were given a certain number of words to identify, while their Electromyographic muscle activity was measured and recorded. The researchers “found high rates of gesturing in narrative speech during clauses with spatial content” [331, p.422]. While the communicative effect of hand gestures, in general, cannot be rejected altogether, the models proposed by authors emphasizing this communicative effect, they seem, nevertheless, to be “best suited for situations in which there are observers who are watching others talking [... rather than] suited to the situation of face-to-face interaction” [238, p.194]. As to the question of *when*, rather than *if*, gestures communicate a message, the answer “depends on a number of factors, including whether the speech they accompany is about motor information, whether the gestures convey additional task-relevant information not present in speech, and whether the beneficiaries of the gestures are children or adults” [221, p.312].

In fact there is a lot of accumulated evidence pointing to the view that that speech and gestures are bound together in linguistic utterance. We have already seen in earlier paragraphs, for instance, that neurobiological and

neurophysiological experimental “data suggest that comprehension of both [spoken and gestural] forms of communication is supported by a common, largely overlapping network of brain regions” although “[t]he neural architecture may differ dramatically at more complex linguistic levels” than those isolated and studied in laboratories [470, p. 20667–20668] (for a more recent overview see [209]). Moreover, it has been proposed that words spoken while executing a gesture, may become one unitary with the aforesaid gesture” [47, p. 189]. Psychological experiments of *delayed auditory feedback* (DAF), that is, “the experience of hearing your own voice continuously echoed back” [318, p. 273], have demonstrated that although the flow of speech during DAF is seriously disrupted, the synchrony between gestures and speech remains virtually intact (p. 273–283). Experiments with blind children have revealed that congenitally blind children *do* gesture during narration, and they “use gesture in the same ways sighted children do” [226, p. 464]; children blind from birth have never seen gesturing and could not possibly imitate them, or use them intentionally as an extra communication channel. A rather intriguing argument comes from research on *phantom limbs*, that is, “the vivid impression that [an amputated] limb is not only still present, but in some cases, painful” [377, p. 1603]. Although originally phantom limbs were viewed as a mental disorder, it was later discovered that limb amputation in adult primates causes massive changes in the cortical organisation of the brain and in the neural representation of the body image within the brain [366] (for a more recent overview see [149, 150]). This post-amputation cortical reorganisation causes phantom limb experiences in the absence of any corresponding sensory data. Moreover, there have been reports of phantom limb patients who experience on a regular basis hand gesturing while talking [376, p. 41], pointing to the fact that “intentions create the sensations of gestures when no motion is possible” [319, p. 244], which can be explained only if we assume that there is a common neural mechanism organising speech and gestures simultaneously.

3.3 The Faces of a Generic Polynomial

One of the interviewees was specialised in algebraic geometry, a branch of mathematics (in fact an independent scientific field today) that tries, or rather began as an attempt, to connect polynomials, their zeros and their geometrical properties. The interview took place in his office, in the university building, and occasionally there were some interruptions. The part of the interview which interests us at this point is when he started explaining what his theory was about. I asked him, in particular, to explain a little bit a part of his theory on Newton polynomials, and then informed him that I knew already some algebraic geometry: algebraic curves, in particular, which I had done a bit as an undergraduate in mathematics. This information was rather crucial in his explanation, because it relieved the interviewee from having to explain to a non expert what is an algebraic curve, and made him focus rather on explaining this to a beginner mathematician. In other words, the effort to use lay metaphors to explain something technical to a non expert was spared, and instead his oral exposition was an explanation to a mathematics student, that is, technically a bit elevated. This kind of question was posed in order to record his technical explanation while he was gesturing. The use of the camera helped capture this coordination between speech and gesture, and the purpose was to try to get a glimpse on the cognitive processes taking place simultaneously with the speech and the gestures. In rather simple terms, the main idea behind talking while gesturing is this: when someone tries to describe to someone else a building which is located away from the place of the interlocutors, then she is using gestures as a method of describing the shape or the architecture of the building; that is common knowledge. The building is absent, it exists though in the interlocutors' imagination, or rather, an imaginary building is being constructed on the spot. If both of the interlocutors are non architects, then their explanation is less detailed, but understandable to each other. If, on the contrary, they are both architects then their explanation is more nuanced and detailed, but still understandable to each other. If one is a lay person and the other an architect, then this poses an extra cognitive burden on the expert, as to how to explain some things to a non expert, and, therefore, what kind of visual metaphors to use.

I was more interested in a dialogue between experts.

The whole exposition, which was similar to a small lecture, was actually divided into four parts, by the interviewee himself, and this was done on the spot. The numbers in square brackets indicate the timer of the computer program⁶ used to watch the interview on the author's computer. The contents of the interview, along with the fully transcribed first two parts of it follow:

Part 1: [40:58–42:23] Short introduction;

Part 2: [42:24–44:56] Gives an example;

Part 3: [44:57–47:12] Describes the big picture of it;

Part 4: [47:13–47:20] Conclusion.

In the transcription that follows the following symbols are used ([319, p. 275–277]):

- / : unfilled speech pause;
- < ... >: filled speech pause;
- # : breath pause;
- * : speaker self-interrupt, self-correction, or restart;
- {...}: uncertain speech transcription;
- % : nonspeech sound.

The interviewer's interventions are denoted by "Me".

Part 1:

[40:58–41:00] So you know maybe <er>

[40:00–41:05] consider generic polynomial a general polynomial of degree n in k variables #

[41:06–41:10] And if you you can do this one thing you have /

[41:11–41:13] polynomial is a sum of monomials

[41:13–41:17] Each monomial {had it all} degree and monomial /

[41:18–41:22] you know degree of monomial is kind of integral point in a lattice

[41:23–41:27] For example you have ex one to kay one so on ex en to kay en

⁶That is *vlc media player*.

Me: Hhm

[41:27–41:30] you can associate an integral point k one k n to it

[41:31–41:37] And polynomial of given degree is just sum of monomials

[41:37–41:41] and monomials corresponds to a point in simplex

Me: Hhm

[41:42–41:46] Some of coordinates are small $\{w_i\}$ and n and all coordinates are not negative $\#$

[41:47–41:50] And integral points in it corresponds to $\#$

[41:51–41:52] correspond to to this equation $\#$

[41:52–41:57] But you can consider for example generic sum of monomials
<er>

[41:58–42:00] some monomials could be missing here $\#$

Me: Yeah

[42:00–42:04] You can consider generic sum of them and it turns out that

[42:05–42:09] to such polynomial the most natural object one can associate

[42:09–42:13] is a convex hull of all these \ast of all these points

[42:13–42:15] this is a polyhedron

Me: Aha

[42:16–42:18] And you know $\#$

[42:18–42:19] so let me show to you

Me: %cough

[42:19–42:22] Just, let me show to you two examples maybe the simplest one

Part 2:

[42:23–42:26] $\#$ Let us consider just polynomial of one variable

[42:26–42:29] this is a stupid example just to get you an idea of what it is about

Me: Hhm

[42:26–42:32] $\#$ Take a polynomial of degree n

Me: Hhm

[42:33–42:34] In one variable $\#$

[42:35–42:42] You have all monomials from degree zero to n so it will be segment Newton polyhedron, will be segment from zero to n

Me: Hhm

[42:43–42:50] # And one can ask well how many solutions equation pee at
ex

[42:50–42:52] ask how many solutions

Me: the usual problem

[42:52–42:54] Yeah the, the usual problem

[42:54–42:57] Yes, and if polynomial is generic

[42:58–42:59] you will have en different roots

[43:00–43:05] If it is not generic some roots can glue together and you will
have multiple roots

Me: Hhm

[43:06–43:10] # But for generic polynomials, for all polynomials but

[43:10–43:14] hypersurface and space of equations number $\{\dots\}$ is the same

[43:14–43:20] and it is equal to en and en is just length of that segment #

Me: Hhm, hhm

[43:20–43:28] okay and let me show to you example which is in fact my theory
and which I would say sounds f* sounds funny

[43:28–43:31] Consider polynomial pee at ex wy is equal to zero #

Me: Hhm [43:31–43:35] It is polynomial two variables and it would defined
a curve

Me: Hhm

[43:36–43:37] Curve

Me: %cough

[43:37–43:42] is, er, a, so, geometric object which one can associate to curve

[43:42–43:47] is this sphere with handles, a number of handles is, is a genus
of that curve

[43:47–43:52] Ok so consider generic polynomial with given Newton polygon

Me: Hhm

[43:52–44:02] One can ask, how many handles this curve will have for generic
equation with given Newton polygon, polygon?

Me: Hhm

[44:03–44:05] # And it turns out that the chance is that:

[44:05–44:11] count how many integral points are they strictly inside this
polygon

[44:11–44:18] And it turns out that number of these points is equal to number

of handles of generic curve

Me: Aha

[44:19–44:21] And so on. So there is a lot of such things

Me: Yeah, yeah

[44:21–44:27] So if you will consider generic system of equations which prescribe a Newton polyhedron

[44:28–44:35] it turns out that basically all discrete invariants from algebraic geometry # can be found

[44:35–44:38] using geometry of these polyhedra.

[44:39–44:41] This geom* by geometry I mean

[44:42–44:51] say volume of this polyhedron mixed volume integral points number in this polyhedron inside this polyhedron in this polyhedron combinatorics of this polyhedron Me: %cough

[44:52–44:55] So all these objects appear in the answers

Most of the interviewees refused to be video recorded. There were nine video interviews recorded, but only three were useful for analysis. The most common reasons were that the camera was not positioned properly, or the recording started in the middle of the technical exposition. Three were rather useful in the end: two in English and one in Russian. For an English-speaking audience the interviews conducted in English would be more useful and insightful, and therefore those two ones conducted in English were selected. From these two interviews one was conducted in a quite technical explanatory way, because the interviewee was convinced that the interviewer was proficient in the subject to be exposed, so he focused exclusively on the exposition of the technical results, rather than the way of how the exposition itself was conducted; finally this interview was chosen to be presented in this thesis. The other, although the interviewee was as well convinced that the interviewer was proficient in the subject, he still presented the subject as if to a lay person with the most probable reason being that his extensive teaching experience was inducing him (unconsciously) to such an expository mode of presentation; still though that presentation could be his everyday mode of exposing, but to get to that deduction more acquaintanceship was necessary than that provided by an hour's interview.

None of the interviewees were informed on the use of the video recorder before the interview. Some of them asked about the interview and its use, and they were informed, among other things, on the use of the video recording *after* the end of the interview. It was necessary during their technical exposition that the interviewees did not know about gesturing and mathematics: had they known that, then their *flux of consciousness*, the main subject of research in this chapter, would have included both the imaginary geometrical objects that the interviewees would be asked upon, as well as their imaginary perception of themselves and of their gestural exposition; this double inclusion would have been expressed in gestural idiom which could create great difficulty in differentiating the geometrical-objects gestures from the self-presentation gestures during gestural analysis.

There have been some other approaches in connecting mathematical concepts and material “blackboard” technology (see [29]) but any type or level of technology, as long there is no artificial intelligence to substitute, even partly, for the human nervous system, human hands will continue to be indispensable, and therefore the only “natural” access points for an ethnographic approach to consciousness and its structures. Even the famous British physicist Stephen Hawking who has a motor neuron disease and is assisted by a speech generating device to communicate with the people around him, uses a special keyboard which scans and detects his intentional, and almost imperceptible, movements of the fingers. Another approach, a *video-ethnography* of classroom mathematics has also been developed quite recently (see for example [191]), but the author considers this method as focusing excessively on materiality that the ethnographer perceives, and not the user himself or herself. Blackboard materiality, for example, acts as a temporary external memory, always available to be mobilised in accordance to cinematic consciousness: it is not an important factor in the formation of mathematical concepts, but rather a material vehicle providing stability. The mathematician’s cinematic consciousness, on the contrary, mobilises imaginary objects, invisible to an ethnographer untrained and uninitiated in obscure mathematical concepts represented materially by mathematical symbols. The mathematically untrained ethnographer’s attention is limited to the observation of material

objects, since that consists representable evidence. What is necessary in this case, as presented later in this section, is to record and examine *not* the materiality of objects, but the detailed *movement of materiality* while it follows the executive commands from the brain in a patterned way. Moreover, in the framework of this thesis, mathematicians, and in general any scientist, does not “form” concepts: a mathematician *sees* concepts, a musician *hears* concepts, a chef *tastes* concepts. The word “concept” itself is misleading as well: it has been borrowed from philosophers whose *source materiality* is rather textual, than visual, auditory, or gustatory.

The gestural analysis procedure to be followed here is McNeill’s method which was as follows: McNeill let the subjects watch a popular cartoon, and then asked them to narrate it to a listener. Then he video-recorded the narration as well as the gestures. So he could connect the cartoon scenes with the gesture. In this thesis a similar method was employed, with one difference. The role of the (material) cartoons was played by research articles. While a cartoon is understandable, or rather visible, by anybody, a research article, along with its imaginary objects, is “visible” only by mathematicians. In order, therefore, to proceed into a *McNeillian* gesture analysis the researcher must have *watched* the *source footage* of the narrative. The procedure for the transcription that was followed was that described in [319, p. 264–272], and referred to as the *Duncan Coding Manual*. In particular, the whole interview was watched without any annotating activity, in order to “develop an initial sense of ‘speaker’ style” [319, p. 264]. Then the interview segment under consideration was transcribed in a very detailed way, “including partials and unintelligibles”, as well as any perceptible sounds involved. Then the speech segment was organised into utterances taking into consideration intonational segments and grammatical syntax rules. Then the gesture phrases were identified in combination with intonation. The kinesic and semantic annotation of gestures was done using mainly McNeill’s division into iconic, metaphoric, deictic, and beat dimension of a gesture [p. 38–44].

The video camera used to take the interviews was a *Flip Video* camcorder by *Cisco Systems Inc.* with a tripod. The software that was used for the analysis and editing of the video recordings were the cross-platform media

player *VLC* by the *VideoLAN* project, the vector graphics editor *Inkscape* by *The Inkscape Team*, and the raster graphics editor *GIMP* by *The GIMP Development Team*. The *VLC* media player can play video with audio at 25% of the original speed, which is important for gesture analysis. The computer operating system on which the editing was conducted was the Linux distribution *Fedora 20* by *Red Hat Inc.* All the software is open-source and can be downloaded by anyone free of charge.

1. [consider **gene**] [ric polynomial **general poly**] [nomial_of_degree **en**] [in **kay** variables #] – [00:01-00:07]

The camcorder during this gesture was not positioned properly and the gesture phrases are visible only if one watches the video.

(A) [consider **gene**]

This is not exactly a gesture; it has only formally been included. The speaker at this point raises his right hand extending his index finger at the same time. The stroke, which is denoted in boldface, is actually a light scratch of the nose with the index finger. In Fig.3.3 we can see that the preparation phase is not visible, due to the position of the camera recorder. The preparation phase takes place between the left square bracket and the beginning of the boldface, that is, just before the stroke takes place.

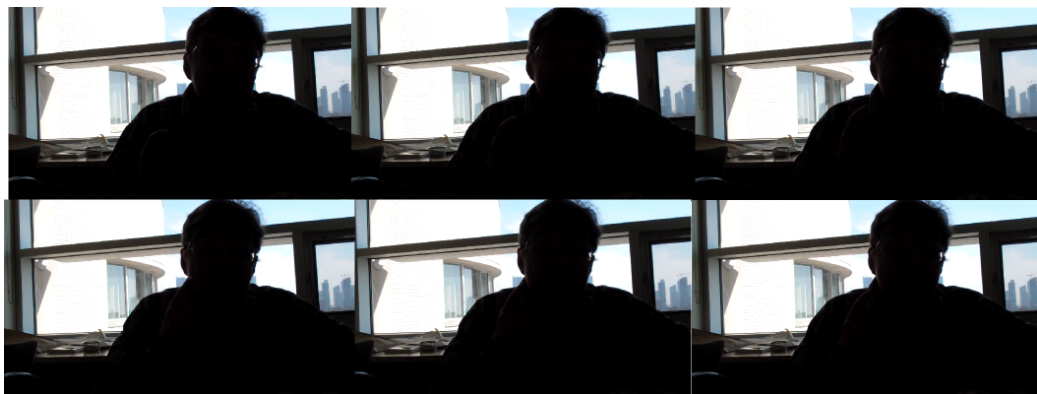


Figure 3.3: Preparation phase of phrase 1A [clockwise].

(B) [ric polyn**omial** general poly]

In the preparation phase, the right arm along with the hand after the index's light scratch of the nose begin to descend towards the desk surface. The palm opens and remains close to the right side of the torso until it touches the desk surface.

When the whole arm touches, but not leaning against, the desk surface, the arm, staying close and parallel to the desk surface, starts moving diagonally along a straight-line trajectory towards the left-hand side of the speaker: this is the point where the gesture stroke begins. The arm remains close to the desk surface, and the palm is open and vertical to the desk surface. The thumb is extended and pointing to the ceiling and the surface of the right-hand palm faces towards the left-hand shoulder as it is moving. Then when the tips of the fingers reach the middle of the front chest area, the arm starts to retreat back to its original position, that is, with the palm close to the right-hand side of the speaker's lower chest. At that point the palm still open becomes vertical while the thumb is still extended and pointing to the ceiling (see Fig.3.4).

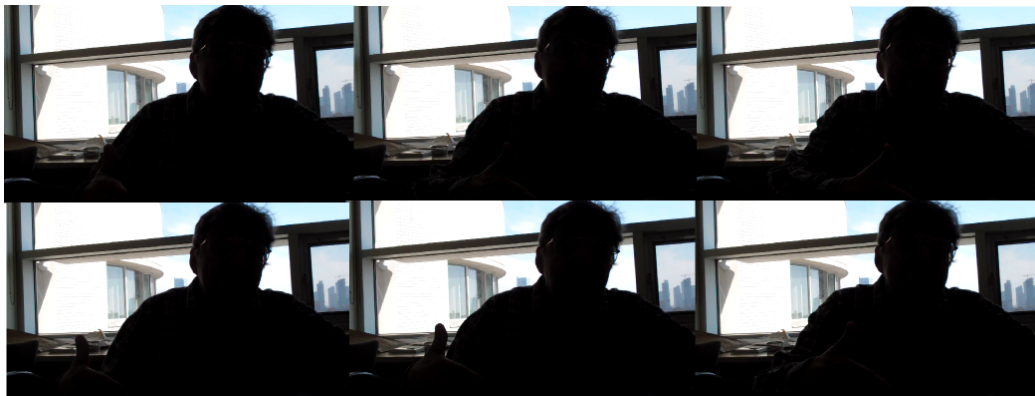


Figure 3.4: Stroke phase of phrase 1B [clockwise].

(C) [nomial _ _ _ _ _ degree en]

During the preparation phase the hand moves a little bit more to the right, while the palm is open and facing the left-hand side of the speaker, vertical to the desk surface, and the thumb is extended pointing to the ceiling (see Fig.3.5). Then there is a momentary hold of the gesture, and

at the stroke phase the outer side of the palm, that is, the side where the little finger lies, touches fast and lightly the desk surface.

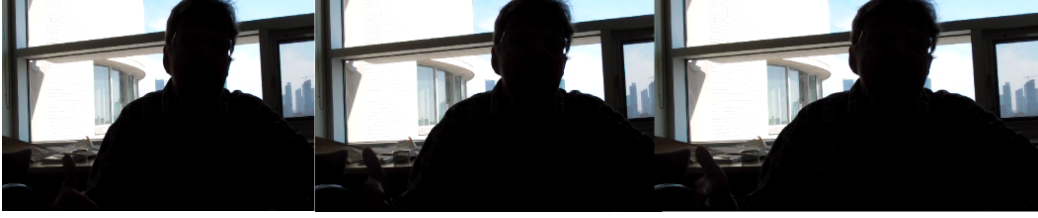


Figure 3.5: Preparation phase of phrase 1C until the hold [left to right].

(D) [in **kay** variables #]

The palm is lifted again, while remaining vertical to the desk surface, and moves a little bit more to the right of the speaker (see Fig.3.6), and then lands and touches lightly again the desk surface while still vertical. The stroke is denoted in bold face. The time interval between the end of the bold face and the right square bracket is the retraction phase of the gesture: the hand moves to a resting position. The retraction phase is not visible because the camera at that moment was being corrected. This denotes, in fact, the end of the semantic content of locution number one.

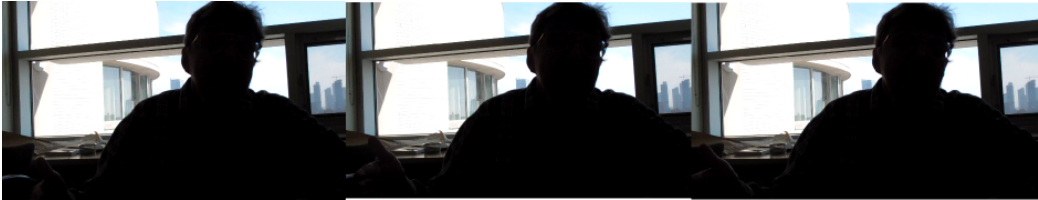


Figure 3.6: Preparation phase of phrase 1D until the right-hand stroke downwards [left to right].

This locution could very well be the beginning of a mathematical theorem. A generic polynomial p of degree $n = 2$ in $k = 3$ variables would be⁷

$$p(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 + x_1 + x_2 + x_3 + 1$$

⁷It is assumed for easier understanding that the coefficients of the variables are all equal to one, as well as the constant coefficient. Otherwise more symbols would have to be employed.

or, in a different form, rather more “generic”,

$$p(x_1, x_2, x_3) = x_1^2 x_2^0 x_3^0 + x_1^1 x_2^1 x_3^0 + x_1^1 x_2^0 x_3^1 + x_1^0 x_2^2 x_3^0 + x_1^0 x_2^1 x_3^1 + x_1^0 x_2^0 x_3^2 \\ + x_1^1 x_2^0 x_3^0 + x_1^0 x_2^1 x_3^0 + x_1^0 x_2^0 x_3^1 + x_1^0 x_2^0 x_3^0,$$

that is, the addition of the power superscripts in each term (i.e. product) of the polynomial is at most 2. The last generic polynomial could also be written as:

$$p(x_1, x_2, x_3) = \sum_{n_1+n_2+n_3 \leq 2} \prod_{i=1}^{i=3} x_i^{n_i}, \quad [n_1, n_2, n_3 = 0, 1, \text{ or } 2],$$

where \sum is the sum of the products \prod . A generic polynomial, therefore, of degree n in k variables would be written as

$$p(x_1, \dots, x_k) = \sum_{n_1+\dots+n_k \leq n} \prod_{i=1}^{i=k} x_i^{n_i}.$$

We can see that the situation with the symbols can easily get out of hand, and the symbols \sum and \prod help to put the limited space of a sheet of paper, or of the blackboard, under some control. The speaker has obviously this in mind, and since a great portion of his career, that is, many hours of his every day life, have been devoted to writing down polynomials, he can easily handle symbols in his mind. However we can still see that he uses gestures to make, or express, his point. Since, so far, the gestures are not conventional, as is, for example, the case with the OK sign when holding the fist tight with the thumb extended, we can say that he uses the gestures according to his *individual idiom*, rather than based on a common gestural code.

It has to be borne in mind, at this point that the speaker is *talking about* generic polynomials, and the listener is *listening about* polynomials. In other words the speaker uses his voice, that is, *sounded air*, to *induce* a certain state of consciousness on the listener, the result of which is the cognitive understanding of the idea of a generic polynomial. The listener can easily be induced into that state of consciousness since he is a former postgraduate student of mathematics. This state of consciousness is usually experienced as memory, which can take, for example, the form of the appearance of mental images accompanied by a sense of profound familiarity. Familiarity with

particular mathematical concepts means that this particular mathematical knowledge has become something very personal, something domestic, something like a daydream on command.

Going back to locution 1, gesture phrase 1A has been included only formally, since it involves a movement of the hand, without any particular meaning related to the locution. Gesture phrase 1B is a very interesting one indeed: the right hand moves diagonally from the right-hand side of the speaker's chest, left and forward to the middle of the front chest area, while the palm remains open and vertical to the desk surface, and the thumb is extended upwards, and then goes back to its original position, close to the right-hand side chest. This hand gesture seems to divide the space into two parts, one from the side of the palm, and the other from the dorsal side of the hand. This is a metaphoric gesture and involves actually the geometry of polynomials: it will become clear in a subsequent locution, where it will be employed again.

Gesture phrases 1C and 1D are beats: the open palm moves down and then slightly up. Their main meaning here is that of a visual-gestural highlighter: they highlight the degree of the polynomial, that is, n , and the number of its variables, that is, k . They also function as punctuation marks, separating these two phrases: 1C functions as a comma, and 1D functions as a full stop.

2. [and if you you you can do this one thing you **have** // _ _/] – [00:08-00:12]

Locution 2, which is only one gesture phrase, comprises four phases. The first phase, the preparation phase, starts when the right hand begins to move from the previous resting position, that is, next to the right left lower chest, towards the centre of the front chest area (see Fig.3.7). Preparation phase happens between the left square bracket and the start of the dotted underline. At the same time the hand forms a fist and the index and thumb fingers are extended and their tips start touching each other. At the end of the preparation phase, and the beginning of the hold phase, the extended index and thumb while touching each other both point to the desk surface. The hold phase is in fact a *virtual hold*: the whole hand, that is, fist as well as index and thumb, shakes repeatedly and very lightly from left to right three

times.⁸ The virtual hold phase is denoted on the text by dotted underline. The stroke phase starts with the boldface and extends over to a pause in speech denoted on the text with a slash “/”. Double slash “//” means a pause while a stroke is taking place. At the beginning of the stroke, the index extends, the thumb folds over the fist and then the finger tip of the extended index touches the desk surface four times. At the end of the stroke phase a proper hold phase starts: the fist is hanging still over the desk surface while the thumb is folded over it and the index is extended and pointing towards the desk surface. The proper hold phase is denoted on the text with dashed underline. The hold continues over the speech pause.

From the start of the preparation phase on, the eye gaze of the speaker is directed to the gesturing of the right hand, until the second proper hold phase; when the second hold starts, the gaze is being directed somewhere to the left area of the front chest space. In other words the speaker during the second hold phase seems to rearrange his exposition. Before the second hold, though, while he was looking down to the desk, the impression he gave off was that he was about to write something. He was looking at an imaginary sheet of paper, at an imaginary polynomial on the desk surface, and he was trying to manipulate it appropriately in order to explain to the listener the polynomials potential properties or its possible algebraic manipulation. In other words, while there was only the empty desk surface present, he was behaving as if something was written on it, or as if he himself could write on it. Since the polynomial was imaginary, that is, there was no material trace of it, as when written with white chalk on a blackboard, for example, the listener could not get a material-symbolic grasp of the speaker’s thoughts. Since there was a continuity in the presentation, actually one cannot be talking about thoughts, as in a plural number of separable pieces of information, but one should be talking about the *flux of the speaker’s consciousness*, which is in fact a cinematic consciousness. The polynomials were all passing by in front of the speaker; they were not visible by a third party because they were not materially visible in a written form. They were, on the other hand, audible,

⁸A *proper* gesture hold occurs when the hand remains still for a few tenths of a second. The frame rate of the video interviews was half a tenth of a second.

due to the speaker's [materially audible] voice, as well as visible as effects of the gesture movements.



Figure 3.7: Preparation phase of locution 2 until the gesture hold [left to right].

3. [**polynomial** is a sum] [**of monomials** #] – [00:12-00:15]

This locution comprises two gesture phrases. The hash sign “#” denotes a breath pause.

(A) [**polynomial** is a sum]

The gesture begins with the stroke phase⁹ The folded fist along with the thumb folded around the fist is dangling above the desk surface twice from left to right, while the index extended follows the fist's movement pointing to the desk surface (see Fig.3.8). The tip of the index finger during the stroke follows a circular-like trajectory. Then the fist retracts to a semi-resting position leaning against the lower front of the right-hand side of the chest. The index remains half-extended during the hold and the little-finger side of the fist touches the desk surface. The hold is a proper hold, that is, the hand is not moving for a few tenths of a second and is denoted on the text by dashed underline.

The gesture is a very close to a beat gesture, since it comprises only of two movements twice. As a beat it highlights the semantic meaning of the gesture phrase, that is, that a polynomial *is* a sum of monomials. A monomial q of degree n in k variables is an algebraic product of the form

$$q(x_1, x_2, \dots, x_k) = x_1^{m_1} x_2^{m_2} \dots x_n^{m_k}, \quad m_1 + m_2 + \dots + m_n = n;$$

⁹The division into gesture phrases is always more or less arbitrary, since there is always continuity of speech; even pauses are a vital element of conversation.

A polynomial, therefore, is a sum of algebraic products of the above form. Note at this point that the speaker quite often does not use the English indefinite article “a”, as in “a polynomial”, that is, a representative of the class of polynomials. This occurs because occasionally he is using Russian syntax. He is not a native speaker of English. In Russian, as well as in Polish and Latin, there is no definite or indefinite grammatical article. For example, in Russian one says “mathematician is writing” meaning both “a mathematician is writing” and “the mathematician is writing”. The accurate meaning is defined by the context.

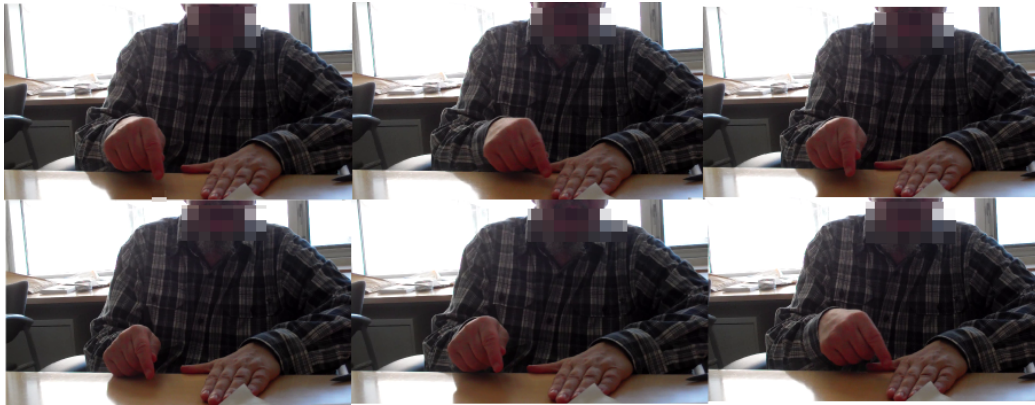


Figure 3.8: Stroke phase of phrase 3A until the gesture hold [clockwise].

(B) [of monomials #]

The phrase begins with the stroke. The loose fist with the index pointing towards the desk surface rotates and at the same time the palm opens up. When the palm points towards the ceiling, with the fingers slightly adducted, the hand moves downwards until its dorsal side touches the desk surface. Then the gesture reaches a resting position (see Fig.3.9).

At this point the presence of the listener enters the scene. This gesture is a communicative one, and it belongs to the so-called *open hand supine* family of gestures.¹⁰ The main characteristic of the palm-up gesture family is that the palm is exposed to the interlocutor, and it has

¹⁰The other gesture family complementary to the Open Hand Supine family is the Open Hand Prone family, that is, palm down (see [239, p.248–283]).



Figure 3.9: Stroke phase of phrase 3B until the gesture hold [clockwise].

been found to be used in some nonhuman primate species, as well. It is used here because it complements the previous highlighting gesture as an explanation of the whole locution, that is, that a polynomial, or rather, any polynomial is a sum of monomials. It is a self-referential gesture which refers back to what has just been said and at the same time implicates the listener as to that reference. While a few seconds ago the speaker was looking at an imaginary polynomial on the desk surface, now he turns his attention to the listener, and therefore the imaginary screen of his cinematic consciousness now projects the listener. At the same time the flux of the speaker's consciousness is not being interrupted at all; it just moves into a new cinematic scene.

4. [each monomial {...had it all} **degree #**] – [00:15-00:17]

This locution consists of one gesture phrase only and ends with a breath, denoted by the hash sign “#”. The breath is included in the stroke. The curly braces with the ellipsis denote unclear speech, and the words in the curly brackets denote an approximation by ear of what that unclear speech could be. During the preparation phase, the right-hand fist is raised from the resting position (see Fig.3.10). As the fist is being raised, the index and thumb fingers are being extended and start touching each other at their tips, while their direction is downwards to the desk surface. Then the finger tips perform a small circle as if holding a pencil and attempting to write something, and start moving downwards to the surface until the adducted

to the fist little finger and then the rest of the adducted fingers as well as the finger tips of the extended and touching each other index and thumb all touch the desk surface. Then, at the stroke, the finger tips of the extended thumb and index,, while leaning against the desk surface, both press hard on the surface.



Figure 3.10: Preparation and stroke phase of phrase 4 [clockwise].

This gesture is again a beat, that is, it emphasises again the importance of the degree of the polynomial. We can also see from the figures of the last gestures, that is locutions 3 and 4, that the left palm is leaning against the desk surface open with the thumb extended in front of the chest area. It is as if the left palm kept the desk surface stable to perform writing on it with the right hand.

5. [**and monomial** ./] [**you know degree** of monomial] [**is kind of integral**]
[**point in a lattice**] – [00:17-00:23]

The gesture here comprises four gesture phrases.

(A) [**and monomial** ./]

The gesture starts with a stroke and ends with a virtual hold, that is, the hand moves almost imperceptibly back and forth over a few tenths of a second and virtually remains still. The virtual hold is denoted by dotted underline. At the beginning of the stroke the folded right fist

starts to rise, while the index and thumb fingers are lightly abducted and touching each other on the tips (see Fig.3.11). Then halfway through the tips of the index and the thumb suddenly change direction and move downwards fast while touching each other, and the whole fist follows suit. Then it stops at about ten centimetres above the desk surface and remains there for a while during the gesture hold. The stroke is fast and lasts about fifteen frames, that is, three quarters of a second.¹¹



Figure 3.11: Stroke phase and then hold of gesture phrase 5A [clockwise].

This gesture is very similar to a beat gesture, and again expresses the emphasis and importance the speaker ascribes to a monomial, which is an important component to a polynomial: it emphasises the importance of it. From what immediately follows on, it appears that the speaker is now thinking about some important property of a monomial that he is about to describe it and connect it to the bigger picture.

(B) [**you know degree** of monomial]

This is another beat gesture which actually clarifies and emphasises the previous locution phrase. At the beginning of the stroke, the finger tip of the index, while the index is extended follows a circle-like trajectory twice upwards and then downwards (see Fig.3.12). The trajectory is first towards the chest, and then away from the chest. The thumb while

¹¹One frame on this particular video interview is one twentieth of a second (0.05 seconds).

extended in the previous gesture phrase, now it is folded on the fist, and at the same time the fist follows the movement of the index finger tip. At the end of the stroke, the index finger remains virtually still above the desk surface in a lateral position. The fist still retains its shape. The whole stroke lasts 1.4 seconds.

This beat gesture again emphasises, or rather draws attention to, the fact that what is to follow briefly concerns the degree of the monomial which is a special number with a particular mathematical property. The movement over space of the index finger tip does not seem to play any role at this point.



Figure 3.12: Stroke phase and gesture phrase 5B until hold [clockwise].

(C) [is kind of integral]

This is a beat gesture performed in a circular movement. At the beginning of the stroke, while the right-hand fist is hanging above the desk surface, and the index finger is abducted in a lateral posture and pointing towards the desk surface (see Fig.3.13), the index finger tip starts to move in a circular way towards the chest, then upwards, and then away from the chest and downwards until it remains still again for a few tenths of a second.

This seems to be a beat gesture, that is, performed in in a two-way movement upwards and downwards, emphasising the fact the the degree of a monomial is an integral point. The circular movement of the finger tip, that is, starting from a certain point above the desk surface, and



Figure 3.13: Stroke phase of gesture phrase 5C until the hold begins [clock-wise].

then returning to a position close to it, after having carved a circular imaginary trajectory, seems to add a certain mapping movement to this beat gesture: starting from the degree of the monomial we are then transferred by means of the index finger tip to an integral point in a lattice. The finger tip in other words is a kind of a pilot who picks up the degree of the monomial and delivers it in a lattice-land on a position of an integral point. This kind of imagery is more common among algebraists: in an algebraic context quite often one talks about *mappings* rather than functions. In Galois theory, for example, if a certain polynomial can be “mapped” onto a certain permutation group, then the corresponding polynomial equation can be solved using radicals.

(D) [point in a lattice]

At the beginning of the stroke, the right-hand fist hangs above the desk surface with the index finger half-extended and pointing to the desk surface (see Fig.3.14). As the stroke starts, the finger tip of the index moves down fast until it touches the desk surface. Meanwhile the index and the fist follow suit and move down. The index finger tip then, after it touches the desk surface, pushes a little bit further the surface downwards. The whole stroke lasts six video frames, that is, 0.3 seconds. Then, when the gesture hold starts, the fist, the extended index finger, remain still for a few tenths of a second while the index finger tip keeps touching the desk surface.

This first thing we can say about this gesture phrase is that it is a



Figure 3.14: Stroke phase of gesture phrase 5D until the beginning of the hold [clockwise].

deictic one, that is, it points to an imaginary geometrical point on the desk surface. That imaginary point on the surface is an *integral point* belonging to a particular algebraic-geometrical set called a *lattice*. A monomial of degree n in k variables

$$p(x_1, \dots, x_k) = x_1^{m_1} x_2^{m_2} \dots x_k^{m_k}, \quad m_1 + m_2 + \dots + m_k = n,$$

is usually written in a multi-index notation in algebraic geometry:

$$p(y) = y^\alpha, \text{ with } y = (x_1, \dots, x_k), \text{ and } \alpha = (m_1, \dots, m_k).$$

The point $\alpha = (m_1, \dots, m_k)$ is actually a point in a k -dimensional space with coordinates m_1, \dots, m_k , and we can see that this particular point lies somewhere on the desk surface, that is, where the index finger point leans against at the end of the stroke. While the discussion is a discussion on a general type of, “generic”, point, this imaginary point, although not materially concrete, as is a point drawn by chalk on a blackboard, its location is a rather concrete one: on the desk surface in front of the speaker. Additionally α is an *integral point*, because its coordinates are integer numbers.

This gesture phrase seems to be a beat gesture as well: while the first movement of the index finger is moving down and touching the surface, the second movement seems to be after the relaxation of the slight push

on the desk surface: the finger moves down, touches the surface and continues moving down, but since the surface is hard and immovable, this continuation of moving down becomes instead pushing down. Then the second move of the beat is the moving back which becomes a relaxation of the push. During the push the extended index seems to slightly bend away from the fist. As a beat, therefore, we can say that it highlights the importance of this integral point being in a lattice.

6. [# **for example you have ex one to kay. one**] [so on ex en to kay en] – [00:23-00:28]

This gesture comprises two gesture phrases and is a very important one.

(A) [# **for example you have ex one to kay. one**]

The gesture starts with a stroke and then goes into a virtual hold denoted by dotted underline. The speaker uses his finger as a pen and actually proceeds into an imaginary writing down of the first term of a monomial that is, as we saw previously. $x_1^{m_1}$. In fact, what he writes down, if one plays back and watches carefully frame to frame the video (one frame in this video is every 0.05 seconds), is x_1^k . During the breathing in, denoted in the text by “#”, the index finger moves lightly away from the chest to the front and lands down on the desk, touching the desk surface with the finger tip, about to start writing, as if the finger itself were a pen (see Fig.3.15, top left screenshot). If we assume that the letter “x” consists of two lateral lines, one slash “/” and one backslash “\” then we can clearly see the following: the speaker first carves an imaginary slash, “/”, that is, the extended right-hand index finger, while the finger tip is touching and sliding on the desk surface, moves from an upper¹² right point on the desk surface, to a lower¹³ left one along a straight-line trajectory [top middle and then top right screen shots in Fig.3.15]; after that the extended index finger raises a little bit above the desk surface and moves a bit to the front away from the chest; then

¹²Upper meaning at a distance from the speaker’s lower chest.

¹³Lower meaning closer to the speaker’s lower chest.

the extended index finger moves down to the desk surface again and it carves an imaginary backslash, “\”, that is, the extended right-hand index finger, while the finger tip is touching and sliding on the desk surface, moves from an upper left point on the desk surface to a lower right one along a straight-line trajectory (bottom right and then bottom middle screenshots in Fig.3.15).

After carving an imaginary backslash “\” and then an imaginary slash “/” crossing each other on the desk surface, that is, creating an imaginary diagonal cross “X”,¹⁴ then the extended index rises from the desk surface, goes to the right lower side of the imaginary “X” and draws an imaginary small vertical line “|”, which is, in fact, the subscript 1 when handwritten. Then the extended right-hand index rises again from the desk surface, the finger tip moves upwards towards the right upper side of the imaginary diagonal cross “X”, and lands there; then it starts carving an imaginary superscript small case “k” (bottom left screenshot in Fig.3.15). If one plays this moment frame by frame, then it emerges that actually the speaker carves an imaginary calligraphic “k” in three strokes of his finger tip, that is, *k* or *k*. The loop of the calligraphic and handwritten letter “k”, in other words, is very evident at this point. A second less probable, but still plausible scenario, is that the third imaginary finger stroke after the “loop” stroke might be the subscript 1 of the superscript k_1 in consonance with the speech locution.

(B) [so on ex en to kay en]

This gesture phrase comprises a stroke and a retraction phase. The retraction phase is very short and it is performed in order to join the left hand in a coordinated movement in the next gesture phrase. After the hold of the previous gesture phrase, that is phrase 6A, the right hand moves down towards the desk surface (Fig.3.16 top left screen shot) and touches lightly the desk surface (top middle screen shot). Then the index finger still extended moves up (top right screen shot), then to the right, and then down again until the finger tip touches the desk surface again (bottom right screen shot). Then again the finger tip moves up (bottom

¹⁴Like the Saltire on the national flag of Scotland.



Figure 3.15: Stroke phase of gesture phrase 6A [clockwise].

middle screen shot), moves further to the right and lands down again to the desk surface (bottom left screen shot), and attempts to write down the last term of the generic monomial: carve an imaginary diagonal cross “X”, then carve a subscript n and then a superscript k_n . This time, if one watches carefully frame by frame the video, the imaginary carvings are not as accurate as in the previous 6A gesture phrase.



Figure 3.16: Stroke phase of gesture phrase 6B [clockwise].

By comparing the audio recording along with the video images of the interview, it is inferred that in this gesture phrase the stroke is a visual-semantic continuation of the previous one: the speaker uses again the extended index as an imaginary pen, and his finger tip as the tip of this imaginary pen. We can also infer that the first two touchdowns of the finger tip are actually the ellipsis in the mathematical formula

which is, in fact, the variables that are missing in the writing down of the monomial for lack of space; he tries, in other words, to write down “ $\dots x_n^{k_n}$ ”, with one dot short during the gesture stroke.

The first thing we can notice in gesture phrases 6A and 6B is that while the speaker talks earlier about a generic monomial of degree n in k variables, he describes verbally its mathematical formula as

$$x_1^{k_1} \dots x_n^{k_n}$$

which, in fact, is a monomial in n variables, and of degree $k_1 + \dots + k_n$. His explanation was an impromptu one, and that would be expected. It raises though a question more important in mathematics: the written mathematical formulas and their memorisation. If the speaker had been writing on a sheet of paper or on the blackboard he would have spotted his mistake and corrected it; in fact this kind of mistake is a very simple one. If, on the contrary, he was writing a very detailed and very long proof of a theorem, such as that provided by the British mathematician Wiles in his proof of Fermat’s Last Theorem, or by the Russian Perelman in his proof of Poincaré’s Conjecture, then the written formulas become much more vital to mathematical proof. The generic polynomial of degree n in k variables, during this interview, is, in fact, a *temporal object*: it cannot be seen or heard, and its existence depends purely on sound voice, and hand gestures; its existence also depends on the memory retention abilities of the interlocutors after the conversation has ended. Memory retention can be conscious, that is many details later can be consciously recalled; or unconscious, that is, few or no details can be recalled later on, apart from the remembrance of the event of conversation.

A generic polynomial in a conversation is elusive as an object in itself. Even a mathematical formula is only a material representation of it. The polynomial can be perceived only in its material modalities: as a written formula, as a geometrical graph, and so on. Moreover, in the case of mathematical innovation, and actually in any kind of innovation, material representations are, at first, absent, and only approximations of them exist. And this what has led

twentieth-century mathematics to an explosion in mathematical innovation: the mathematician proposes new entities; in the case of polynomial algebra an example of these new entities were the “algebraic varieties”; then he or she writes down some formula he or she thinks will improve his or her imagination, and then corrects himself or herself. At the time of invention, one could suggest, materiality plays a secondary role: imagination has taken control, and the formulas have only become imagination’s material prosthesis. At this moment the only available objective means for a sociologist of science to observe the process of innovation are only the mathematician’s gestures, his or verbal descriptions, and the way he or she handles the relevant materiality, such as written formulas. The new mathematical object is temporal, it occupies the flux of the mathematician’s cinematic consciousness, and changes imaginary form all the time. The final innovation product is in fact not the beginning or the middle of innovation processes but the *end* of it: that is the moment when the imaginary reality acquires a material cloak and becomes continuously visible.

7. [# **you can associate**] [an integral **point kay one kay en**] [to it] – [00:28-00:31]

This gesture consists of three gesture phrases.

(A) [# **you can associate**]

During the brief retraction of the right-hand index finger in the previous gesture phrase 6B, the right-hand extended index finger to rises up from the right to the left, while the left hand palm starts to rise from left to right and approaches the left-hand index finger (Fig.3.17, top left screenshot). The preparation phase of the gesture phrase 7A occurs during the breathing in, denoted in the transcription by the hash sign “#”. When the right-hand extended index finger, along with the right-hand fist, reaches a high point in the front right-hand higher chest area, it starts to descend towards the desk surface, while the left hand palm in a reverse cup formation hovers above the desk surface (Fig.3.17 top middle screenshot). Then the extended right-hand index finger continues to descend, the left-hand palm, still in a reverse cup formation, starts

to descend until they both touch the desk surface (Fig.3.17, top right screenshot). Then the left-hand palm, still in a reverse cup formation, remains still, while the right-hand performs a gesture. The stroke of the right hand consists of three parts: the index finger after the touchdown begins to carve with the finger tip on the desk surface an imaginary left parenthesis “(” (Fig.3.17, top right screenshot); then the finger tip rises again and moves over the desk surface to the right for a few frames (Fig.3.17 bottom right screenshot), and then lands down again on the desk surface. Then the finger tip, after the touchdown starts sliding on the desk surface forming the trajectory of a right parenthesis “)” (Fig.3.17, bottom middle screenshot). When the right-parenthesis trajectory finishes the finger tip stops there for a few screen shots.¹⁵ While the right-hand index finger is standing on the finger tip, which, in its turn, leans against the desk surface, the right-hand fist begins to push down, with the effect of causing the extended index finger to slightly, almost imperceptibly, to bend (Fig.3.17 bottom left screenshot). Then the preparation phase of the next gesture phrase begins.



Figure 3.17: Gesture phrase 7A [clockwise].

The fact that the two hands start moving means that they had received instructions from the brain to coordinate. In other words, the meaning of the particular gesture phrase demanded more cognitive energy to be consumed, since coordination of hands means increased attention. That

¹⁵The time span between two screen shots, which is actually a video frame, lasts for 0.05 seconds in the video of the interview.

alone would be enough to demonstrate the importance of the meaning of this gesture phrase. Moreover, the stroke could actually be analysed into three substrokes. First the carving of an imaginary left parenthesis “(”, then the carving of an imaginary right parenthesis “)”, and then the light push of the right index finger on the desk surface. The first two strokes, as well many of the strokes in the previous gesture phrases, would be iconic, since they imitate the writing of the two parentheses, to form an ordered n -tuple in the next gesture phrase. The interesting thing, though, is that since this demonstration was impromptu, that is, unplanned, the gestures actually have to do mostly with how the speaker organised his thoughts spontaneously, rather than following a plan. The carving of the parentheses means that the parentheses were *indeed* on the desk surface; their material form was the hand movement, which the listener was in a position to follow, since himself had mathematical training. If there was no relevant training, probably this would have eluded him. The hand movement of the speaker, in other words, was the material cinematic substratum of the imaginary public sphere of the conversation. Both the speaker and the listener could *envision a common spirit world*, unavailable to the uninitiated. They were momentarily both shamans sharing a common state of consciousness, which could be called a common knowledge. If this had been a presentation of a more advanced mathematical topic, then this common spirit world most probably have been more exclusive.

The third stroke, is actually a stroke of a beat gesture, because it consists of two movements: one downwards, that is the pushing down, and one upwards, which is the relaxation of the pushing. As a stroke of a potential beat gesture it highlights the semantic content of the locution phrase, that is, the connection of the degree of a polynomial with an integral point in a lattice.

(B) [an integral **point kay one kay en**]

During the preparation phase the extended right-hand index finger rises and begins to move towards the left-hand side of the speaker (Fig.3.18, top far left screen shot). Then it stops in front of the central chest area

above the desk surface (Fig.3.18 top middle left screenshot), and it starts moving down towards the desk surface (Fig.3.18 top middle right screen shot). Then the finger tip remains hovering above the desk surface for a few frames, and then it moves down until it touches the screen surface (Fig.3.18 top far right screenshot). Then the finger tip rises slightly above the desk surface, moves a bit to the right-hand side of the speaker while in the air (Fig.3.18 bottom far right screen shot), and then lands again on the desk surface (Fig.3.18 bottom middle right screenshot). And then once more the finger tip of the right-hand extended index rises above the desk surface (Fig.3.18 bottom middle left screenshot), moves a bit further to the right-hand side of the speaker and then lands again on the desk surface. The left palm still in a reverse cup formation remains on the desk surface and the left arm is positioned in a diagonal direction with the left-hand ankle next to the lower left-hand side chest area, and the reverse-cup palm touching the desk surface in front of the lower middle chest area.



Figure 3.18: Gesture phrase 7B [clockwise].

This gesture phrase is a continuation of the previous one. After having carved an imaginary left and then a right parentheses the speaker now populates the parenthesis pair with numbers. What he does, in other words, is putting the coordinates of an integral point in an ordered n -tuple, that is, if the monomial is written as $x_1^{k_1} \dots x_n^{k_n}$, he inserts the coordinates k_1, \dots, k_n into the previously written imaginary parentheses (), and through an imaginary montage of the just previously written parentheses and the light touches between the parentheses a new imaginary temporal product has just been created: the ordered n -tuple

(k_1, \dots, k_n) . The ordered n -tuple when written down has a certain spatial ordering of symbols:

$$“(”, “k_1”, “\dots”, “k_n”, “)”$$

The flux of consciousness, therefore, based on its short-term memory of the listener, or rather the watcher, after having formed a mental projector slide of the parentheses () projects upon it the next mental projector slide of the ordered coordinates k_1, \dots, k_n and forms a new imaginary slide, the n -tuple (k_1, \dots, k_n) . A real slide on a projector is transparent. In a similar way the desk surface is transparent, or rather invisible. What matters is the voice and the hand gestures. In that particular moment of the interview the hand gestures were not that visible to the watcher, but they were visible to the speaker; in fact the hand gestures were following the instructions of the mental projector slides of an imaginary blackboard or an imaginary sheet of paper with written formulas and words. In fact, in long mathematical proofs, for example those with more than thirty pages, it is almost impossible, in the author's experience, to understand the proof without this kind of imaginary montage. Many proofs conducted on a blackboard demand the erasure of the first steps of the proof, in order to write later ones, because of lack of space. The students write down notes in order to keep a record of the steps. But still notes cannot be easily spread out over a surface next to one another, and the reader still has to turn pages, and then turn back again to check what he or she has forgotten.

(C) [to **it**]

This gesture phrase consists of a preparation phase and a stroke phase, and both of the hands are involved in it. The preparation phase begins with the lifting up of both hands: the right-hand extended index finger starts to rise up diagonally upwards towards the centre of the front lower chest area, while the left palm still in reverse cup formation begins to rise vertically (Fig.3.19, far left and middle left screenshot). Then, just before the stroke, the right-hand extended index finger along with the fist, as well as the left palm stop above the desk surface simultaneously

(Fig.3.19, middle right screenshot). The right-hand extended index finger is pointing now downwards to the desk surface while the palm is hanging parallel above the desk surface with the left-hand fingers adducted together and pointing down to the desktop, and the left-hand thumb slightly extended and bent pointing to the left. Then the stroke begins: the right-hand extended index finger moves down to the desk surface and the finger tip touches the desk surface, while the left-hand palm remains still in its previous position and bent with the fingers adducted and pointing to the desk surface (Fig.3.19, far right screenshot).



Figure 3.19: Gesture phrase 7C [left to right].

The fact that both hands coordinate, and therefore more cognitive effort is exerted, betrays the importance that the speaker accords to the semantic content of the locution: that the degree of a monomial corresponds to an integral point. If a monomial is written in the multi-index notation

$$y^\alpha = x_1^{k_1} \dots x_n^{k_n}, \text{ where } y = (x_1, \dots, x_n), \text{ and } \alpha = (k_1, \dots, k_n),$$

then the degree $k_1 + \dots + k_n$ of the monomial corresponds (is *mapped onto*) an integral point, that is, a point on a Cartesian coordinates system with integer coordinates. The gesture is a deictic gesture, that is, it is pointing to the integral point somewhere on the desk surface, or more specifically to the touchdown point of the right-hand extended index finger tip. In other words, that particular imaginary “integral point” is located, on the desk surface. The desk surface, therefore, plays in fact the role of an imaginary blackboard, or of an imaginary blank sheet of paper. The flux of the speaker’s consciousness has assimilated for good now the desk surface as a necessary imaginary performance prop equipment to complete the mathematical exposition. The desk

surface has now become *transparent*, that is, a prosthetic extension of consciousness and therefore invisible.

8. [# and # polynomial of given] [degree is just //] [sum of mono] – [00:32-00:38]

This gesture phrase consists of three gesture phrases. The third gesture phrase comprises only a stroke phase.

(A) [# and # polynomial of given]

The hash signs denote breath-ins. During the preparation phase the wrists, both the left and the right one, start rising, the palm start forming a cup formation facing towards the chest. As the palm continue to rise the fingers of one palm, while adducted meet on the air the adducted fingers of the other palm (Fig.3.20, top left screenshot). Then during the stroke they continue to rise simultaneously while the adducted fingers of one palm touch the dorsal side of the adducted fingers of the other



Figure 3.20: Gesture phrase 8A [clockwise].

palm for a few frames, and then they start moving away from each other (Fig.3.20 top middle screenshot). At the same time both of the palms start descending, while open with the fingers adducted; the palms now are in a diagonal position facing each other, while their dorsal sides are facing towards the desk surface (Fig.3.20, top right screenshot). Then both of the palms, still in a diagonal position and facing each other and

upwards start moving ahead until the tips of the adducted fingers on one palm came close to the tips of the adducted fingers of the other palm above the desk surface (Fig.3.20, bottom right screenshot). Then the palms withdraw backwards towards the chest (Fig.3.20, bottom middle screenshot). At the end of the backward withdrawal the palms are open, facing each other and in a vertical position with respect to the desk surface, the fingers are extended and adducted, and the thumbs are extended and pointing to the upwards; at that point the palms start descending until their little-finger sides touch the desk surface (Fig.3.20, bottom left screenshot). Then the palms remain in this position during the hold phase of the gesture phrase.

This gesture is again a member of the family of Open Hand Supine (“palms-up” family), that is, a communicative one. Its meaning is that the speaker operates at the moment under an explanatory mode with respect to the semantic content of the locution. At the same time it has serious indications of a deictic gesture: both of the vertical palms during their forward movement component of the stroke end up with extended and adducted fingers very close, almost meeting each other above the desk surface (Fig.3.20, bottom right screenshot). This forward movement seems to point to the degree, that is, the sum of the exponents, of the imaginary monomial that was written a few seconds ago on the desk surface

(B) [degree is just //]

This gesture phrase consists of a preparation phase and a stroke, which occurs during a pause. A stroke during a pause is denoted on the text by a double slash “//”. At the beginning of the preparation phase, the right palm starts to rise upwards, while the the left palm starts to withdraw backwards towards the lower left chest area, just above the left front area of the upper abdomen (Fig.3.21, top left screenshot). Then the right hand forms a fist, the thumb is folded on the fist and the index finger extends and points down to the desk surface, while at the same time the left palm starts hanging on the desk surface edge from the index from the index, the middle and the ring fingers (Fig.3.21,

top middle screenshot). At the beginning of the stroke, which is the beginning of the pause in the speech as well, the extended right-hand index finger starts moving towards the left of the speaker (Fig.3.21, top right screenshot). Then it stops, the index finger tip follows a circular trajectory twice, then moves back and forth, and finally jumps into the stroke of the next gesture phrase.



Figure 3.21: Gesture phrase 8B [clockwise].

This gesture actually has no meaning, or rather has no meaning related to the speaker's speech. It has been named by McNeill a "Butterworth, after the British Brian Butterworth who mentioned these kind of gestures as speech failures (see [73] and [318, p. 77]). What happens at this point is that the speaker is momentarily reorganising mentally the exposition, or maybe he saw something irrelevant to the exposition, but relevant to his attention, and the listener missed, or whatever. In the terminology of this thesis, the speaker's flux of consciousness "tripped over", or came across "a mental road bump", and as a result, the hand could not follow a clear neural executive order from the brain, or rather it was receiving many contradictory executive orders. So, in fact, this gesture stroke was more of a *gestural pause*, rather than a gestural clause.

(C) [sum of mono]

After the stroke of the previous "gesture pause", during which only the index finger is extended and the other right-hand fingers are folded into a fist, the next gesture stroke immediately follows. At the beginning

of the new stroke the other right-hand fingers join the extended right-hand index finger and they all form a bent palm facing the speaker's chest (Fig.3.22, far left screenshot). Then the whole right-hand palm starts to rotate around the wrist: the right-hand finger tips move down in a circular trajectory, then move upwards towards the chest (Fig.3.22, middle left screenshot); then finger tips continue their rotation around the wrist upwards until they reach the apex of their trajectory; then continuing their circular trajectory start descending and moving away from the speaker's chest, with the palm facing upwards (Fig.3.22, middle right screenshot); finally the open palm with the fingers adducted stops right above the desk surface with its face up and its dorsal side facing the desk surface (Fig.3.22).



Figure 3.22: Gesture phrase 8C [left to right].

Again this gesture phrase is a “palms up” gesture: the speaker is in an explanatory mode, that is, in a communicative mode, rather than in a manual-constructive mode as in many of previous gesture phrases. During the communicative mode the listener and probably the listener's reactions enter the flux of the speaker's consciousness. This phrase has only a stroke phase, and after the stroke the hand moves immediately to the preparation phase of the next gesture phrase.

9. [mials and monomials **corresponds to a point**] [# in a simplex] – [00:38-00:42]

This gesture consists of two gesture phrases and it will be analysed in a more detailed way than the previous ones: it is most probably the most revealing gesture in this whole interview segment presented here.

(A) [mials and monomials **corresponds to a point**] Gesture phrase 9A consists of a preparation phase, a gesture hold phase, and a gesture

stroke phase. We left the last gesture phrase with the right-hand palm hovering above the desk surface: the palm slightly bent is open and faces the ceiling, the fingers are adducted, and the dorsal side of the hand faces downwards (Fig.3.23, top left screenshot). And the preparation phase begins. The open right-hand palm starts to rotate around the wrist, with the axis of rotation being the along the palm; it rotates clockwise from the point of view of the listener, counterclockwise, form the point of view of the speaker (Fig.3.23, top middle screenshot).



Figure 3.23: Preparation phase of gesture phrase 9A [clockwise].

In other words, during the hand rotation, the open palm initially faces the ceiling, then faces the office wall on the left-hand side of the speaker, and the rotation ends up with the palm half bent and passing into a fist formation facing the desk surface (Fig.3.23 top right screenshot). During the time that the right hand rotates and simultaneously is passing from an open palm formation into a fist formation, the index finger starts to extend pointing downwards, while the rest of the fingers are beginning to fold over the face of the lower palm (Fig.3.23, bottom right screenshot). When the rotation is complete the right hand is in a fist formation with the index finger extended and pointing to the desk surface (Fig.3.23, bottom middle screenshot). Then the right-hand index finger tip moves a bit towards the left-hand side of the speaker and at the same time descends a few centimetres towards the desk surface (Fig.3.23, bottom left screenshot). At this point the gesture hold begins, which lasts for 34 frames, that is, for 1.7 seconds, during which the whole hand remains

almost still.



Figure 3.24: First stage of stroke phase of gesture phrase 9A [clockwise].

The stroke phase of the gesture phrase will be divided into two stages (Fig.3.24 and 3.25). After the gesture hold, during which the right-hand extended index finger tip hovers above the desk surface away from the speaker's chest, the finger tip of the right-hand extended index finger moves down and touches lightly the desk surface (Fig.3.24, top far left screenshot). Then the finger tip rises, moves closer to the speaker's chest (Fig.3.24, top middle left screenshot) and then descends again and touches lightly the desk surface (Fig.3.24, top middle right screenshot). Then the finger tip rises for a second time and moves above the desk surface closer to the speaker (Fig.3.24, top far right screenshot) and descends and touches the desk surface for a third time (Fig.3.24, bottom far right screenshot). Then the finger tip rises for a third time above the desk surface, moves towards the speaker's chest (Fig.3.24, bottom middle right screenshot), and then descends for a fourth time and touches the desk surface (Fig.3.24, bottom middle left screenshot). Just when the second stage of the stroke is about to begin, the middle and the ring fingers of the left palm which was hanging all the time from the edge of the desk make their appearance (Fig.3.24, bottom far left screenshot).



Figure 3.25: Second stage of stroke phase of gesture phrase 9A [left to write].

At the beginning of the second stage of the gesture stroke the right hand in fist formation hovers above the desk surface in front of the central chest area of the speaker, the right-hand index finger is extended pointing to the desk surface, and the finger tip of the right-hand index touches the desk surface (Fig.3.25, far left screenshot). Then the whole right-hand fist keeping formation starts moving along the edge of the desk to the right-hand side of the speaker in front of the speaker's chest: the right-hand fist crosses the front chest area of the speaker from left to right, while at the same time the right-hand extended index is touching the tip of the desk edge, and the right-hand index finger tip is sliding along the tip of the edge desk (Fig.3.25, middle left, and then middle right screenshots). At the end of the second stage of the stroke the right-hand index finger tip stops almost imperceptibly above the edge of the desk surface close to the left lower chest area of the speaker, while the fist hovers above the desk surface, and the right-hand index finger still remains extended (Fig.3.25, far right screenshot).

During the first stage of the gesture stroke the extended right-hand index finger touches the desk surface four times successively while at the same time the speaker talks about the points to which a monomial correspond: in other words these light touches are actually the places where these imaginary points are located. On the blackboard the points become visible by the trace a chalk leaves on the surface. At the moment of the interview, though, a point becomes visible through hand, palm and finger movements. During the second stage of the gesture stroke we see the extended index finger of the right hand carving actually a straight line, which considering what preceded and what follows, seems to be the x -axis of the Cartesian coordinates, or at least a reference axis. This axis is imaginary, but visible because of the sliding of the tip of the extended right-hand index finger. At the same time this axis is visible to someone with an elementary training in geometry, a state of consciousness that is induced mainly by the speaker's voice. A little child could never have imagined these without actual drawing of the straight line on a blackboard. Considering that the interaction under consideration is part of an interview on spaces of arbitrary dimension, that is $1, 2, \dots, n, n +$

1, ..., these particular gestures are in fact incomprehensible to a lay audience, especially to an audience without any training in algebraic spaces. These potentially incomprehensible hand gestures, though, are a sociologist's access to a mathematician's "spirit world."

(B) [# in a simplex]

This gesture phrase consists of a preparation phase, which is denoted by normal typeface between the left square bracket and the beginning of the boldface fonts, of a stroke phase, denoted by boldface fonts, and of a retraction phase, denoted by normal typeface between the end of the boldface fonts and the right square bracket. The preparation phase happens during breathing in.



Figure 3.26: Gesture phrase 9B [clockwise].

At the beginning of the preparation phase the right-hand fist hovers above the desk surface, the right-hand index finger is extended and pointing downwards to the desk surface (Fig.3.26, top far left screenshot). During the preparation phase the fist opens up above the desk surface and changes into an open palm which is vertical to the desk surface and facing the office wall on the left-hand side of the speaker (Fig.3.26, top middle left screenshot). Then the stroke phase is performed in three stages. First the thumb is extended and abducted while the open palm preserving orientation moves forward into a straight-line trajectory (Fig.3.26, top middle right and then far right screenshots). At half an arm's distance away from the speaker's chest, the right-hand palm rotates clockwise with respect to the listener, and counterclockwise with respect to the speaker, and starts facing downwards to the

desk surface (Fig.3.26, bottom far right screenshot). Then the palm moves towards the speaker's body closing in on the central upper chest area (Fig.3.26, bottom middle right screenshot). Finally the open palm diagonally positioned and (still) facing downwards with the thumb (still) extended and abducted, moves away from the speaker's chest while at the same time descending towards the desk surface in a chop-like movement and stops for a few frames above the desk surface (Fig.3.26, bottom middle left screenshot). Then during the retraction phase, the open right-hand palm descends and touches the desk surface on its little-finger side in a diagonal position, with the thumb finger (still) extended and abducted, and remains there on hold, until the preparation phase of the next gesture.

The interesting point here is the meaning of the word simplex. A simplex is a generalisation in n -dimensions of a triangle or a cube. A triangle or a square are two-dimensional and the triangle has three edges, and the square has four edges. A tetrahedron is a three-dimensional pyramid with four faces, with each of its faces is a triangle, while a cube is a three-dimensional geometrical solid with six faces. The tetrahedron is the three-dimensional geometrical object which corresponds roughly to the two-dimensional triangle, while the cube is a three-dimensional geometrical object which corresponds to the two-dimensional square. The two-dimensional faces of the tetrahedron and the cube correspond to the one-dimensional edges of the triangle and the square; and they all have angles. The triangle and the square are *two-dimensional simplices*,¹⁶ while the tetrahedron and the cube are *three-dimensional simplices*. In a similar way, using algebraic definitions, n -dimensional simplices can be defined. One well known simplex in geometry is the *hypercube*, that is, the four-dimensional cube. Having that in mind, what one can infer, is that actually the stroke phase of this gesture phrase is a gestural representation of two of the faces of the n -dimensional simplex in consideration at the moment. The first face of the simplex is during the first stage of the stroke (Fig.3.26 top middle left, then top middle right, and

¹⁶Plural of *simplex*.

then top far right screenshots), and the second face of the simplex are the second and the third stages of the stroke (Fig.3.26, bottom far right, then bottom middle right, and finally bottom middle left screenshots). In other words, momentarily an imaginary n -dimensional simplex was present between the interviewee and the interviewer, and it had all been constructed by the interviewee himself on the spot. It was not visible, though, in the same way to each one of them. Since that simplex was in fact a temporal object, that is, it had no permanent material trace, after a few seconds, when the talk moved into other topics, that simplex dissolved like smoke in a windy day.

3.4 A Multitude of Worlds in Everyday Life

There has been recently an increasing interest in the relationship between hand gesturing, mathematics understanding and communication in the classroom. Besides McNeill's groundbreaking research, another major impetus was cognitive research on creativity. For quite long the human brain had been considered modular, that is, the brain was assumed to be divided into *brain modules*, each one specialising in certain functions. Quite recently, though, this view has changed, as we saw in the second chapter. One major defining characteristic between the early *Homo* species and modern *Homo sapiens* seems to have been a newly evolved *cross-modular* brain architecture. According to Steven Mithen, the early *Homo* species brain was like a Romanesque cathedral which had compartmentalised chapels: every module was like a separate chapel specialised in its functions without communicating with other chapels. Modern humans, on the other hand, evolved a brain more like a Gothic cathedral with

a new architectural feature: direct access between the chapels [i.e. brain modules]. With this feature, knowledge once trapped within different chapels can now be integrated together. It is not quite clear how this direct access was achieved. [325, p. 76].

The major conceptual mistake, in other words, that many archaeologists (as well as the majority of social scientists one could add) until very recently had been making, in Mithen's opinion, was that they assumed that

the Early Human mind was just like the modern mind – that there was a *cognitive fluidity* between social, technical and natural history intelligences. We can only make sense of the archaeological record, and solve the puzzles we have found, by recognizing that these were isolated from each other. Just as there was a cognitive barrier between technical and natural history intelligence, so too were there barriers between these and social intelligence [325, p. 154, italics added].

Cognitive fluidity, though, means more “fluid” artefacts, as well as, more “fluid” materiality. In the case of *hand* gestures during *sound* discourse while explaining *geometrical* concepts we can say that the brain modules processing *kinesthetic*, *auditory*, as well as *spatial* information communicate internally and produce a new more “fluid” mathematical artefact called simplex, as we saw in the interview presented in the previous section.

The “fluidity” of mathematical artefacts has taken various directions in recent research. Tall has presented the view that pupils, and later students, acquire mathematical thinking by engaging during their studies with

“three worlds of mathematics” – the “conceptual-embodied” world based on perception, action and thought experiment, the “proceptual-symbolic” world of calculation and algebraic manipulation compressing processes such as counting into concepts such as number, and the “axiomatic-formal” world of set-theoretic concept definitions and mathematical proof. Each “world” has its own sequence of development and its own forms of proof that may be blended together to give a rich variety of ways of thinking mathematically[433, p. 5].

Hand gesturing, according to Tall, takes place in the “conceptual-embodied” world of mathematical practice. Fauconnier and Turner have proposed the

idea of *mental spaces*, that is,

small conceptual packets constructed as we think and talk, for purposes of local understanding and action. They are very partial assemblies containing elements, structured by frames and cognitive models. [...] [I]n terms of processing, elements in mental spaces correspond to activated neuronal assemblies and linking between elements corresponds to some kind of neurobiological binding, such as co-activation. [145, p.102].

During everyday human interaction these mental spaces induced by natural and social context become *conceptually integrated*, or *conceptually blended*, which “is an invisible, unconscious activity involved in every aspect of human life” [145, p. 18]. In the case of hand gesturing while explaining mathematics, for example, *grounded blends* “result from the blending of elements from a mental space with elements of one’s immediate physical environment” [289, p.283]. When explaining, for example in a calculus course, the notion of anti-derivative, the *physical gesture space*, that is,

the space in front of the body in which gestures are typically performed, which usually extends vertically from the waist to the eyes and horizontally between the shoulders [473, p.374],

becomes “endowed with *mathematical* meaning” [473, p.377, italics in the original].

In mathematics education gestures, in general, play an important role: they help the pupil in learning counting [4], facilitate the knowledge of numbers [68], mediate in explaining and understanding graphs [50, 339, 375, 385], as well as the derivative of a function [22]. Gestures, in other words, have come to

be considered as a part of individuals’ *sensuous attempts* at dealing with abstract cultural ideas. Gestures may be seen as part of

one of the *sensuous modes* – the tactile mode – which is demonstrated in the efforts at conceptually grasping something [375, p. 115, italics added].

When a pupil, for example, is being explained a mathematical concept, such as the derivative or the antiderivative of a function, the employment of gestures, along with voice, graphs and written symbols, during class instruction can lead to

the creation of *virtual mathematical constructs* – multimodal constructs that are created via sensuous cognition using gestures, speech and other related semiotic systems [474, p. 891, italics in the original].

Arzarello, expanding on the semiotic approach of mathematics learning (see, for example, [135, 374]), has described the various communicative modalities in class instruction, such as speech, written symbols, gestures, and so on, as *semiotic resources* and has furthered this theoretical approach even more by defining the concepts of a *semiotic set*, and a *semiotic bundle*:

A *semiotic set* is: (a) A set of signs which may possibly be produced with different actions that have an intentional character, such as uttering, speaking, writing, drawing, gesticulating, handling an artefact. (b) A set of modes for producing signs and possibly transforming them; such modes can possibly be rules or algorithms but can also be more flexible action or production modes used by the subject. (c) A set of relationships among these signs and their meanings embodied in an underlying meaning structure. [...]

[Moreover, a] *semiotic bundle* is: (i) A collection of semiotic sets. (ii) A set of relationships between the sets of the bundle. Some of the relationships may have conversion modes between them [21, p. 281].

When a teacher engages, usually unconsciously, as well as consciously, in such a semiotic bundle with the students during instruction he or she participates in, what Arzarello has called, a *semiotic game*:

he *coordinates* with the semiotic resources used by the students and then *guides* the development of knowledge using these resources. Typically, the teacher uses the same gestures as the students and rephrases their sentences using precise mathematical language. Doing so, he supports the students towards a correct scientific meaning [22, p. 106, italics added].

The previous paragraphs are very reminiscent of Roman Ingarden's aesthetic theory whose "contribution to the philosophy of art remains unmatched – and virtually unknown, especially among Anglo-American philosophers" [113, p. 365]. According to Ingarden, a *work of art* has a multi-layered existence. A literary work of art, for example,

is stratiform, possessing at least four strata: (1) the stratum of word sounds; (2) the stratum of language meanings (the first and second forming the bistratum of language); (3) the stratum of perspectives in which the objects and situations represented in the work appear; (4) the stratum of these objects and situations. Once finished, the work of art forms a whole, but its parts are so ordered that it unfolds while read in time; this is called the quasi-temporal structure of the work of art [170, p. 13–14].

The strata of the book itself, i.e. the bound paper pages, as well the printed letters on the, usually, white pages precede those strata already mentioned for the literary work of art. A sculptural work of art, on the contrary,

is a form with two and sometimes three heterogeneous strata: (1) the stratum of schematic aspects, (2) the stratum of the three-dimensional form of represented objects or situations, and (3) the stratum of literary or historical content, whenever present [364, p. 261].

The work of art, in other words, is not identical to a real object as perceived by the senses, but it is an *aesthetic object*, *cognized* in a gradual process over time: it is an *act of consciousness*, rather than a real-material object; it is a “*change of attitude* from a practical one, assumed in everyday life, or from an investigating one, to an aesthetic attitude” [223, p. 295, italics in the original]. The aesthetic object, in other words, is the product of the aesthetic experience of the reader of a novel, the viewer of a sculpture or a painting, or the listener, in the case of music.

Ingarden went further on and included dance, theatre, and cinema as aesthetic objects. At this point it would be more appropriate to swap Ingarden’s term “work of art” for the term “artefact” as used in this thesis with one caveat, though: artefacts, in contradistinction to works of the (fine) arts, one could argue, bear almost no sign of Walter Benjamin’s *aura* which

registers the *irreducible specificity* or *uniqueness* of the traditional art object. It derives from the origin of art in ritual. In modernity, art is characterized by the destruction and decay of the aura from technical [as well as from massive-industrial] reproduction [355, p. 37, italics added].

The social semantics of an artwork’s aura, in other words, would refer to “an elusive phenomenal substance, ether, or halo that surrounds a person or object of perception, encapsulating their individuality and authenticity” [64, p. 340]. Apart, therefore, from the aura that surrounds a work of art, and differentiates it from being a generic artefact, we can formulate the working assumption that aesthetic perception of an artefact is quite similar to aesthetic perception of a work of art. If we include in the list of artefacts dance, theatrical plays, and cinematic films, we can, then, grasp much better what an artefact, in fact, is: an *experience* rather, than a single material detachable object. The prehistoric stone tool now becomes the human body which is put under a disciplinary regime both *consciously* and *with a plan* in mind. King Lear, the Shakespearean King Lear, that one sees in a theatrical play, is an *embodiment*, or *incarnation* if you like, of King Lear, and not the concrete person of King Lear. Martha Graham’s choreography and dancing technique

did not arise accidentally but it was the the result of conscious design and laborious effort in a similar fashion to the construction of a handaxe. And who could not accept that Marcel Marceau, the famous French mime, was very “verbose” with his “silent” narrative pantomime? What defines an artefact, in the end of the day, is not the material from which an artefact is made, or its final form, but the *nervous system*, and especially the *human brain*, which organises the *information received* from the sensory neurons, and *arranges the construction* of the artefact into its final form.

The artwork which is the closest one in its aesthetic structure to mathematics teaching and classroom instruction is the dramatic artwork. According to Ingarden the dramatic work of art is aesthetically structured in four ontological, or existential, if you like, steps [326, p. 166]:

- Step 1 *The written work*: The author composes the main text (text a) and the side text (text b). Each text consists of undetermined objectivities that exist as potentially determined.
- Step 2 *The stage play*: The director, actors, et al. actualize the potential objectivities of text b and, in so doing, partially actualize the potentialities of text a.
- Step 3 *The performance*: The actors further actualize the potentialities of text a.
- Step 4 *The aesthetic object*: The spectator further actualizes the potentialities of text a.

Starting with the first step in the aesthetic experience of class instruction we can identify the main text of the “instruction” artwork as the material to be taught, e.g. the derivative, or antiderivative, of a function, and the side text, which is usually not written down, as the way that the material will be taught. An “instruction” dramatic play on the derivative of a function is, usually, one of many plays in a “theatrical series” called “Calculus”. Each play has a certain number of *characters*, or rather, following Paul Ricoeur, a number of *quasi-characters* in the “Calculus narrative”:

nothing in the notion of character, understood in the sense of someone who performs an action, requires that this character be an individual human being. [...] [T]he role of character can be held by *whomever* or *whatever* is designated in the narrative as the grammatical subject of an action predicate in the basic narrative sentence “X does R” [388, p. 197, italics in the original].

Since a derivative is not a human, or rather it is known by the audience not to be human, it cannot be impersonated in front of the class; other means have therefore to be employed to “embody” it in a more “impersonal” way, such as a blackboard graph or hand gestures. The graph, or the set of gestures, to be employed are part of the side text of the “instruction” play, that is, how the characters are to be presented and impersonated, and not of the main text, that is, of the way the story is presented to an audience. Since a derivative is not human it cannot engage into dialogue, and, therefore, has to be narrated using the voice and the body of the instructor along with the appropriate props. The authors of the main text of the instruction theatrical play are mathematicians of the past, as well as the authors of the textbook being used in the classroom. The author of the side text of the instruction play is, usually, the instructor himself or herself. The side text can become a side text proper when the instructor has written down notes on how to present the material.

Moving on to the second step of a dramatic artwork we can say that in class instruction the director, as well as one of the main actors is the instructor himself, or herself. He is one of the main actors as a narrator, rather than the protagonist. Moreover, as one of the actors he or she makes himself, or herself, appropriately presentable to the audience as the social role of an instructor dictates. The social role of an instructor is part of the side text of the instruction play, but usually not written down; it has usually been acquired through personal experience. At this step the instructor-director gathers the appropriate props for the instruction play, such as chalks, projectors, slides, and so on. The third step is the execution of the theatrical instruction performance itself. The quasi-characters take up their role in the narrative, and the instructor-narrator using hand gestures, his or her voice, his or her

body, as well as other objects as props brings to life quasi-characters such as a derivative. What is interesting, as well as the major difference between a traditional theatrical play, and the instruction theatrical play is that the instructor-narrator breaks the *fourth wall*, that is, he or she not only talks to the audience, but also encourages to audience to participate; the audience in other works participates quite actively in the plot as assistant directors. Getting to the fourth step, the step of reaching the aesthetic object, we reach the level where the derivative of a function has acquired, or at least it has been attempted to acquire, a status similar to the Shakespearean King Lear: an imaginary entity engaging the (cognitive) attention of the audience as a real king in his own flesh and blood would have done.

Going back to the previous section and seeing how the interviewee was explaining what an integral point is we have some interesting observations to make. Gesture 7, as it was demonstrated, consists of three gesture phrases: 7A, during which the speaker uses his index as an imaginary pen to write down on the desk surface an imaginary left-hand-side and then a right-hand-side parentheses, that is, “(” & “)” ; 7B during which he populates the parentheses with the imaginary symbols “ k_1 ”, “...”, and “ k_n ”; and 7C during which he points with his extended index to these imaginary populated parentheses on the desk surface. The interviewee was sitting on a swivel chair behind his desk, and the only theatrical props he was using to present his arguments were the desk, his hands, and his voice. But then some new props appeared as the explanation was unfolding: first the imaginary parentheses, and then the imaginary letters and the ellipsis. How can we say that these were props? Simply because the third gesture phrase was pointing to them. There is though another reason for that. When in classroom an ordered n-tuple (k_1, \dots, k_n) is written on the blackboard, so that the students see it, and not just hear it. It is written because the instructor wants to keep it visible for some time during the class instruction, so that the students can go back to seeing it any time on the spot. So both the interviewee, and the interviewer during the gesture both knew what was that written down on the surface table: it was an invisible to an uninitiated viewer. The act of using the index finger as an imaginary pen or an imaginary chalk, would

be difficult to understand, as well, for someone who had been illiterate, and had never written, or had never seen other people write. The pointing to that imaginary prop in gesture phrase 7C means that the ordered n-tuple (k_1, \dots, k_n) had become an aesthetic object visible only to those initiated into polynomial algebra. In other words, although these particular props of mathematical symbols were imaginary, they still directed joint attention along all gesture phrases 7A, 7B, and 7C.

The use of the desk as a paper page or blackboard substitute was the use of an object the presence of which was rather stable: an earthquake or an explosion were most probably the only events that could destabilise the presence of the desk surface during the interview. The voice, on the other hand, as well as the hand gestures were rather of unstable presence and therefore the employment of memory was a necessity. The interviewee's hands, as three-dimensional visual objects were on the move most of the time, while the voice was changing pronunciation and intonation constantly; both changes depended on whether the interviewee wanted to emphasize, point, illustrate his point and so on. Gestures, as well as voice, in other words were *temporal* artefacts, or rather, temporal material props functioning as *pipelines of consciousness*: they helped both the speaker and the listener to elevate their consciousnesses to a new perception of *particular* (aesthetic) mathematical objects. During gesture 7 three consecutive gestures occurred: 7A, 7B, and 7C. In the beginning, during 7A, a pair of two imaginary parentheses was written with an imaginary blank space between them. After a few seconds the blank space was populated with numbered letters (letters with subscripts), and an ellipsis. In other words 7B was *added onto* 7A, forming an imaginary ordered pair, i.e. k_1, \dots, k_n . After 7B the speaker, as well as the listener, cannot see 7A as separate, but only 7A and 7B together. i.e. as an ordered pair. Then 7C “points” to the imaginary ordered pair. In other words, although there are no mathematical symbols written on the desk surface, the imaginary ordered pair gradually becomes “visible”. While cognitive attention is being directed towards the ordered pair, it is being directed towards an imaginary cluster of artefacts, i.e. written mathematical symbols. The order pair has now become, in Ingarden's terminology, a “purely intentional object”, which,

as Ingarden has always emphasised in his works, is an “act of consciousness”, rather than a real object. This particular ordered pair, towards the end of the gesture phrase 7C, is becoming an (Ingardenian) aesthetic object which has been *concretised* in the particular context of that particular interview: “[t]he outcome of the process of the aesthetic *concretization* of the work of art is the aesthetic object” [429, p. 24, italics added].

A very important property of human consciousness is *intentionality* which is

that property of many mental states and events by which they are directed at or about or of objects and states of affairs in the world [405, p. 1].

The word *intentionality* is “derived from the Latin word *intentio*, which meant, roughly, ‘having an idea’ or ‘the directing of attention in thought’” [299, p. 1]. When Ingarden, therefore, refers to an aesthetic object as a *purely intentional object*, what he means is that the aesthetic object is a result of “mental states and events” of consciousness, whether that object exists, or is a figment of one’s imagination. This thesis refers to mathematical objects as products of *altered states of consciousness* to emphasize the fact, that mathematical objects are imaginary, rather than real-material, and that they are so due to the extensive training a professional mathematician has been put through in order to be able to perceive the imaginative ends of a mathematical proof. Gesture phrases 7A, 7B, and 7C, in other words, refer to aesthetic objects, rather than real. They can be easily considered as real while they are being presented in this thesis for a very simple reason: the reader of this thesis, and in fact any reader, as a very well literate person has been schooled for years, if not for decades, and therefore, he or she will automatically, that is, as a reflex, will “see” these mathematical ordered pairs. The imaginary symbols in gesture phrases 7A and 7B are, therefore, momentary products of the *altered states of consciousness* of each reader or viewer of the video interview, but as such they are *forgotten* to be so.

Bernard Stiegler, initially following Heidegger, distinguishes between the *who*, that is, the user of the tool, and the *what*, that is, the tool being used which is the tool as ready-to-hand:

The *who* is opposed to the *what* in that it has hands, being itself neither present-at-hand [i.e. held but not being used] nor ready-to-hand [i.e. already being used]. Having hands, it has *whats* [sic] present at and ready to hand. This *what* that the hand handles makes up a system. It is a “technical system” that completely saturates the world [421, p. 244, italics in the original].

During the interview, a part of which was presented in the previous section, both the interviewer and the interviewee were quite competent in understanding what a polynomial, monomial, as well as a convex hull, and so on, are about: the interviewer as a former student in a mathematics department; the interviewee as an innovator himself. During the presentation of polynomial theory the interviewer had a personal interest in polynomial theory apart from the sociological interest dictated by his doctoral research. The interviewee, on the other hand, had a personal interest because, as it unfolded during the interview, and not just as a logical conclusion, it was a chance for him to present his own personal contribution: when asked, in other words, on his own theory, that was a *personal* question, rather than a formal and impersonal one. As the part of the interview on the exposition of polynomial theory unfolded over time, at some point time itself was *disappearing* and the whole “world” at these particular minutes was becoming “saturated” by polynomials. Although the hands and hand gestures of the interviewee at the start of the exposition were visible to the interviewer, as objects as well as subjects of research for the interviewer, at some point they disappeared, and polynomials started to appear on the spot: consciousness of the interviewer, in other words, *withdrew* to the next *layer of existence*, to a world of *purely intentional objects*, as Ingarden would say. The hands and voice, in other words, as tools, that is, as material vehicles of meaning, were gradually disappearing, and the craftsman himself, that is, a craftsman of his own bodily movements and voice generation, was gradually becoming *transparent*: the distinction between the *who*, and the *what* was becoming quite unclear. Hand gestures, and voice intonations, in other words, during the exposition were, in Stiegler’s terminology, instances of *tertiary retention*, that is, a third type of memory, as opposed to primary retention, or short-term memory,

and secondary retention, or long-term memory. Tertiary memory is a memory type which needs artefacts to become activated and starts functioning by becoming *forgotten* as an external memory support:

A tool is, before anything else, memory: if this were not the case, it could never function as a reference of significance. [...] The tool refers in principle to an already-there, to a fore-having [sic] of something that the who has not itself necessarily lived, but which comes under it in its concern. ... A tool functions first as image-consciousness. This constitutivity of “tertiary memory” grounds the irreducible neutrality of the *who* – its **programmatically**, including above all the grammar governing any language [421, p. 254–255, italics in the original, teletype font added].

Stiegler borrowed the idea of the *programmatically of tertiary memory* from the archaeologist André Leroi-Gourhan’s concept of *programs* as activities of animal intelligence, back in the 60s:

the nervous system is not an instinct-producing machine but one that responds to internal and external demands by designing *programs* [281, p. 221, italics added].

Human intelligence, in Leroi-Gourhan’s view, with respect to artefact construction and their use, goes beyond the biological imperatives of genetic code:

A program emerges from the contact between the nervous system of an individual organism and the contingent stimuli of a specific environment. ... There is thus no originary and inescapable program laid down by genetic code, because programs are “constructed” en route, so to speak, in the existential process of confronting situations and actualizing a “disposition” under conditions that cannot be envisioned in advance [345, p. 100].

Human speech and hand gestures during particular social contexts, following Stiegler, as material assemblages of auditory and visual artefacts, are, in fact,

a complex set of programs “constructed en route” which, though, do follow a certain syntax as to their “rules of engagement”. As it was demonstrated in the second chapter, there is an increasing awareness among the archaeological, cognitive, and artificial intelligence communities that language, tool use, and hand gestures are more deeply related, than previously thought, and the direction now is toward their common neurological substratum: language, in other words, is not just a medium for social construction, as the majority in the STS community seems to hold, but artefact use and hand gesturing *are* social construction proper. One such research trend, for example, is the research on *action grammars*, that is discovering the underlying syntax of human activities such as tool use, or hand gesturing (see, for example, [194, 354]). Speaking a language, such as English, French, or Chinese, *is already a technical skill* acquired during one’s childhood: it can take years to learn a foreign language, and sometimes linguistic competence equal to that of a native speaker can never be accomplished in one’s life.

Harry Collins, in his attempt to redirect the research STS has proposed new definitions for expertise, taking into consideration the level of expertise. *Interactional expertise*, according to Collins,

allows that the *language* of a community whose members are embodied in one way can be acquired by individuals with bodies that are shaped differently and in ways that prevent them from participating fully in the physical activities of that community [102, p. 80, italics added].

Collins then defined the *minimal embodiment thesis*, according to which

[al]though bodily form gives rise to the *language* of a community, only the minimal bodily requirements necessary to learn any *language* are necessary to learn the language of any community in which the organism is embedded [102, p. 79, italics added].

The emphasis of interactional expertise is placed on language, and as language is considered the sound version of it, and not in its totality, as we

saw earlier. Although the author of this thesis would totally agree with Collins's call for a new research direction in STS, he still considers Collins's approach rather flawed in some respects. The most important has to do with how Collins implicitly defines language, that is, as speech, rather than as a more general engagement with materiality, which is the underlying definition in the implicit consensus in the archaeological, neurobiological and cognitive science communities. The notion of interactional expertise needs to be broadened, this thesis argues from language as speech to language as gesturing, as well as artefact construction. Moreover, speech, in particular, has such an important position in all human cultures because of another reason, in the author's opinion: the abundance of material resources for speech. Any artefact needs a certain amount of resources to be constructed. Hand gestures, for example, need hands to create visible movement, and human speech needs atmospheric air with a certain percentage of oxygen and nitrogen as well as a very well developed respiratory system to create voice. Every diver who goes to great depths and uses oxygen and helium instead of oxygen and nitrogen to breathe air knows that the human voice is quite different in that case when helium and oxygen are exhaled. Creating the Parthenon, on the contrary, or a skyscraper, poses certain limitations on artefact construction.

Collins's definition of interactional expertise, in the author's view, seems to be an attempt to domesticate materiality for the purposes of data generation in social research: in the end of the day a main method of data collection in the science studies is *audio recordings*, or in the case of this thesis *video recordings*. Moreover, the culinary arts, or even the perfume industry, have in general eluded the attention of the STS community, although innovation in that area has been tremendous in the last three decades; it can be indeed quite difficult to create audio or video recordings of gustatory or olfactory artefacts and concepts. Let us not forget that both Collins, as well as, all the well known and academically successful social scientists are not famous only because of their innovations: they are famous because of their high competence in using a purely auditory in its origins, and slightly visual in its use, artefact to communicate their innovations: the *discursive scientific article*. The social scientific community is a *systemically closed* community, that is, it

cannot accept cognitive stimulation from its social surroundings unless it has materially translated these social surroundings into its own institutional semantics, and the main material resource for this institutional translation are its monetary resources which, as we will see in the next chapter, in the case of science, are the (discursive) scientific journal and the (discursive) scientific article. In the next chapter, we will see how the Soviet mathematical social system, in fact, as a special case of a global scientific system was socially embedded in the Soviet society.

Chapter 4

The Banks are Always to Blame

4.1 Introduction

The open source movement in computer programming is considered today fully legitimate (see for example [148]). The idea behind open source is to share the computational knowledge, rather than hide it from the programmers' community. When a new computer programme is written, the programmer writes it in a certain, programming language, such as C, C++, Java, or Python. Most of the Facebook social networking service, for instance, has been written in PHP, a programming language especially designed for internet applications, while some parts of it have been written in C++, for very high performance. The part of a computer programme that has been written in a programming language is called the *source code*. Then the source code is translated into *machine code*, that is, simply put ones and zeros that the particular computer machine can execute. This translation is carried out by another programme, designed especially for the computer machine that the programmer is using, which can be *a compiler*, when the translation is performed separately from execution,¹ or *an interpreter*, when the translation is carried out at the same time as the execution of the programme.² When the programmer who wrote the original source code publishes it online, and shares it with the community, then the programme is *open source*, otherwise it is *closed sourced*. Big and well known companies, like Microsoft Corporation, or Apple Incorporated, are using the closed-source model, because they base their business models on copyright laws and intellectual property to generate income. They sell their programming applications (*apps*) and they charge their customers for these. Closed source programming, on the contrary, is scoffed at by the hackers' various communities and advocates of free software because knowledge should be free. We should not be thinking, in the case of open source, of "free as in free beer... [but] free as in free speech" ([459, p.132]).

A closer look, in fact, reveals that software programming has a lot in common with mathematics. If the source code of a programme is written in

¹*Compile time* is separate from *runtime*.

²Compile time is during runtime.

an appropriate programming language comprehensible to the programmer, then the source code of a mathematical theorem is written in a human language along with the appropriate mathematical symbols comprehensible to the mathematician. The machine code of a programme is “comprehensible” to the particular computer machine; the machine code of a theorem is “comprehensible” to the particular brain of the mathematician proving *ad hoc* the theorem. When a mathematician executes the proof of a theorem he or she picks up a pen or a pencil and starts writing on a sheet (usually) of paper; there are of course the archetypal images of ancient Greek mathematicians drawing geometrical figures on sand with a wooden stick. When a programmer formulates a programme he or she writes it on the computer screen using some special programme for that, like a text editor or a word processor, and executes the programme with the use of another third programme, that is a compiler, or an interpreter.³ Since in mathematics the processes of theorem formulation and proof execution are not separate, the mathematician can much easier pre-experience the execution through his or her imagination. In software engineering, on the contrary, due to this separation of programme formulation and programme execution, pre-experiencing is quite difficult, and almost impossible in large software projects. The whole phase of checking the validity, execution and performance of a software project is called *software testing* which is part of the *software development life-cycle* (for more on these see [300]). The equivalent of software testing in mathematics is conducted in the form of peer-reviewing of articles during which the validity of a proof is “executed” by brains specialised in the corresponding field, i.e. by experts on that particular discipline.

When a hacker writes a programme it is mainly because “the pure joy of craftsmanship is the primary motivation” [379, p.82]. Technical excellence in programming is in high demand, actually, since today’s programming is very complicated, and large businesses such as Google, or Oracle Corporation count their assets in tens of billions in dollars: they cannot rely only on people who work only for profit; long hours of work time demand intrinsic

³Actually it is much more complicated than that, and the whole process in software engineering is called *software building*(see [418]).

motivation, that is, enjoying programming as a hobby at the same time. The situation is no different in mathematics, especially in fields such as algebraic geometry, or functional analysis, which are both very demanding in terms of intellectual effort, and by common conception they would be considered as useless in everyday real life in terms of applications; of all the mathematicians teaching at the university level whom the author has personally met, including the interviewees for the present thesis, not a single one has ever said that mathematics is boring; on the contrary, they were all deeply interested in their fields and the neighbouring ones.⁴ Moreover, all the hobbyist hackers, and the professional mathematicians, as well, share a common custom: not only do they enjoy their hobbyist or professional activity, but they want to *give it away*. Open source movement is actually based on the idea that by publishing online the source code of a programme and sharing it with the rest of the community improves both the programme and the community. It would be beyond imagination, in the case of mathematical physics, if Einstein had kept his theories to himself, by keeping his manuscripts private. In fact, in both programming and mathematics there is actually an underlying *gift culture*, since “success in gift exchange becomes a matter of giving away as much wealth as possible, so as to gain a social advantage” [180, p. 36]. And the most obvious social advantage in science can be social status. What is more interesting is that gift economies “arise in populations that do not have significant material-scarcity problems with survival goods” [379, p. 81]: the Pythagorean Theorem is an abundant intangible asset; it is available to anybody, even to those who cannot decipher it. Gift-economy narratives formulated and presented by anthropological studies on various indigenous peoples would actually seem to be the most appropriate initial models to describe the economy of science. Is the number of published scientific articles *not* a display of an author’s wealth within that particular scientific discipline?

⁴There where, of course, occasional complaints such as low salary, or less interested people, etc.

4.2 The Gifts of Science

Part of the fieldwork conducted for this thesis were face-to-face interviews with Russian mathematicians. A great portion of the questions concerned their personal biographies as related to their profession as researchers. Although there were many expected commonalities to be found, one, rather unexpected, began to surface. In communist Russia, as would be expected, it was important for a university student to have a kind of an unofficial patron: some very well established professor who would help the talented student to find employment in some mathematical, or at least mathematically oriented, institute or establishment. In the case of the Lomonosov University, for example, Kolmogorov for a few decades was such a patron. But during the seventies, there were many such patrons and factions began to form: some subjects were more popular than others, some new subjects were introduced, and some attracted more students than other ones. They could be called schools, as well, as in philosophical schools, but the word faction seems more appropriate, because the students and the professors who were “members” were emotionally quite close. There was, of course, always the problem with the approval of the communist party and its leadership, but in the case of mathematics this did not affect the developments in research: “Whatever Stalin said, one plus one is always two,” as one of the interviewees half-seriously and half-jokingly remarked. In one noted exception, Kolmogorov had tried to expand his mathematical interests to cybernetics and biology, but his research was cut short rather abruptly, because biology was a rather dangerous topic in Stalin’s period [167, p.56–61]. Nevertheless, these “research factions” were not actually competing with each other, because the Soviet Academic of Sciences, the Division of Physico-Mathematical sciences, was providing ample support to any new mathematical research endeavour that was proposed by any of its senior member. There were though persecutions of Jews and implemented especially during the Stalinist period (see for example [360, 361]). This persecution policy had an effect on the mathematical community as well. Lev Pontryagin, for example, one of the most influential mathematicians of the 20th century, who became blind at the age of 14, was repeatedly accused for anti-Semitism to the point that he felt

obliged to defend himself against such accusations publicly on the *Science Magazine* in front of an international academic audience [367].

After Stalin's death these persecutions were officially banished from the political agenda, but their impact was still strongly felt, as the case with Pontryagin had showed. Many of the interviewees in the United States were of Jewish descent and they stated that they felt discriminated against during their professional lives in the Soviet Union. It became a difficult task for some of them to find employment after the submission of their candidate's thesis,⁵ and they ascribed it to a general antisemitic policy of higher ranks in the academic establishment. Their employment by well known American universities is evidence that their work was indeed considered by their North American peers as important, but in the end of the day it is indeed very difficult to establish what exactly had happened in each individual case, except only by police investigation; but this lies outside the realm of this thesis. Nevertheless, what emerged from interviews was that some mathematicians who found it almost impossible to be employed by research institutions specialising in some branch of mathematics, finally did manage to find a job, in institutes irrelevant to their specialty. In one such case, the interviewee

“found a job in [a] power institute... It was a huge institute [with] about 1000-1500 researchers. I still don't know what they were doing... It was a part of Soviet Union life. So, they were supposed to do kind of research in power production, distribution. Nobody cared about anything, so it was a waste of time. So I formally was in this power institute for 15 years... For 15 years my job was just to sit, but luckily I was able to continue to do my research.”

He then went on to describe the difficulties of publishing articles in his case, because he needed a clearance from the institute: that was meant to prevent state secrets from being published. And when the head of a power institute has to approve an article, for example, on “quantum deformation” or on

⁵The candidate's thesis was, and still is, equivalent to the the doctoral dissertation in English-speaking universities.

“rational varieties”,⁶ it is not in the best of circumstances for the author of that article. Sometimes he published his articles as an appendix to another author’s article, in which case the article author was officially responsible for the content of the whole article. During many of his working days he would go to his office, sit and write his research papers. This pattern of research activity, that is, employed by an institution irrelevant to the employee’s official training, was found in about 20% of the interviews, and some official CVs posted online.

If we focus our attention for a minute on a scientific article, that is, the fact of being a scientific article, we can consider the author of it as its owner, its proprietor; its author, or authors, do consider it as his or her article anyway. And, especially in mathematics, as in any other scientific discipline, the author is always mentioned, on the header. No article is being published without mention of its author or authors. And an author in mathematics, especially in twentieth-century, has gone through extensive training for at least seven to eight years; in the case of the Soviet Union considering the existence of mathematical specialised schools, that training rises up to 9-10 years. So a published article is indeed quite significant, given the human labour that has been invested. And herein bursts into the scene an unexpected, or rather neglected, aspect of mathematics in the Soviet Union: the *gift economy* of scientific mathematics. The Russian mathematician of the previous paragraph, who worked in the power institute for fifteen years, whose job responsibilities were vague both to him, and to many around him, instead of giving up research, decided in the opposite direction: he decided to *donate* his articles, to devote his personal labour to the pursue of his research interests without any visible benefit. His salary, as everybody’s salary in Soviet times, was too low to be considered as a motivation to excel, and he did not consider it himself at the time. And this was in general the mentality that emerged during the interviews in general: they were all eager to participate in *a system of redundant transactions* [89, p.19], that is publish articles, without the expectation of drastically improving their social standing, or receiving

⁶If the reader does not know what these mean then he or she could feel much more accurately what the head of that power institute would have felt.

any other apparent economic benefit whatsoever. Today, for example, the situation in English-speaking academia is quite different: there is spectacular funding involved, especially in subdisciplines of high demand such as financial mathematics, or business administration. Publishing articles in a specialty of high commercial demand has become a business enterprise now, and cannot be considered any more only as a system of gift economy.⁷ Even Kolmogorov, who had the world reputation and the social standing any working mathematician would envy, and had no apparent reason to pursue any more research, was so productive, that after probability theory he moved into mathematical logic and algorithmic theory and published in these fields articles of substantial importance. Today in computer science a computability measure bears his surname, that is, “Kolmogorov complexity.”

One could of course argue that publishing an article has future expected rewards for the author. So the author’s peers could not consider it as a gift, that is, as the product of “genuine” scientific motivation. In some interviews there was indeed mention of prestige, respect by peers, and even ambition. And that indeed is a plausible counterargument: they are not actually gifts, but investments. It can be argued actually that those mathematicians working in research institutes other than their specialty were working there in the hope of getting a better post in the future in a proper mathematical institution. And a job in such an institution had the advantage, for instance, of travelling abroad to international conferences, meet foreign scientists, scientists from “capitalist countries” as they were referred to in the official reports of the Steklov Mathematical Institute, or meet other countries from the Eastern Bloc, that is, scientists from “democratic countries.” Economic explanations of profit maximisation are indeed plausible as accounts of personal motivation. There were many benefits actually of acquiring a job in a mathematical institute for the researcher to think about. It has to be borne in mind, though, that if the submission of an article can be considered an economic transaction, then it is not barter: there is no simultaneous exchange of some goods, and the parameter of time is involved. And besides that “[g]ift

⁷In Britain, actually, due to the Research Assessment Exercise publishing articles has become compulsory.

exchange is an exchange in and by which the agents strive to conceal the objective truth of the exchange, i.e. the calculation which guarantees the equity of the exchange” [60, p.22]. A gift, in other words, in order to be considered a gift, it has to be presented as such, that is, as the outcome of generosity, or, in the case of scientific articles, as the outcome of intellectual interest and scientific curiosity. The time between the exchange of gifts and counter-gifts is what makes them gifts, and the existence of this time period ensures that the official pretense of the pursuit of pure knowledge, and not of profit, is observed. And in science, in particular, the time that elapses from the publication of an article the moment it pays back various benefits can be a rather long period to wait for an “investment” to pay off few gains, not to mention the limited gains of a communist country when compared to the Western ones. Nevertheless, an article can indeed be considered as a gift for one more reason: due to the time that intervenes between its publication and its potential future benefits, it provides, at least in the case of Soviet Union it did so, the pretense of a genuine scientific motivation.

One of the author’s favourite theorems in functional analysis is the *Gelfand-Naimark*⁸ *Theorem* [166]. What it says, although indeed intriguing, is not as important as another aspect of it. Israel Gelfand and Mark Naimark, two of the most important Soviet mathematicians, published their article in 1943. It is a very fundamental theorem in modern quantum probability and some aspects of quantum mechanics: it established actually overnight the very fascinating and intricate field of *C*-algebras*.⁹ In 1993 a book was published by the American Mathematical Society to celebrate the fiftieth anniversary of *C*-algebras* since the first publication of the Gelfand-Naimark theorem (see [127]). This volume is actually the proceedings of a conference on *C*-algebras*. The first chapter of it is the republication of the original article with some minor corrections. So from a lay persons point of view, there are still people dealing with something written some 50 years ago. It could be said, as well, that the Bible is much older. But here is the interesting aspect: just as the Bible is the gift of God to His people, the Gelfand-Naimark

⁸Old spelling “Neumark.”

⁹Pronounced C-star algebras.

theorem is Gelfand's and Naimark's gift to mathematics, or rather, to the future students of mathematics. The original article of 1943 has acquired a special name, that is *Gelfand-Naimark theorem* and has acquired a special property that some special artefacts possess: it has become an *inalienable possession* [456]. Neither Gelfand, nor Naimark can be any more separated from the gift they themselves gave away, even now that they have passed away. It can be said that indeed their memory has remained in the hands of posterity, but in fact another aspect is more interesting: the act of giving away a gift when still in life. A king, for example, can donate his crown to his son, or a father bequeath his surname to his children. Insignia of rank can be given away, as well, but they still belong to an abstract authority which has accorded them. Gelfand and Naimark gave away one of their gifts, but it became something more: it became their common *heirloom*; it is not a usual gift any more.

Any article, therefore, published in a mathematical journal, especially a very well known one, from the moment it acquires the name of the original author, becomes a donated heirloom. And this is actually very close to recognition from peers. Recognition in mathematics, in terms of cultural anthropology, is achieved when a mathematician's heirloom becomes, not only publicly accepted, but also *well known and widely shared* within the community of peers. This is actually another aspect of immortality, or in a rather more Marxian terminology, another aspect of the mode of self-production of mathematics, and science in general. If we were talking about Homer's Troy and Achilles' feats, that would be one of the many "[e]xamples of heroic ventures through which individuals strive for immortality in efforts to deny death" [456, p. 7]. But we are talking about something more mundane: the leading personages of a scientific profession. They were not the descendants of the Olympian gods, neither were they demigods to possess superhuman abilities, nor were they cult leaders: they were the victims of the Weberian disenchantment of the world and rationalisation of society. But they did accomplish important feats that intrigued their academic descendants. And when we see first Gelfand's and Naimark's original article, and then the book celebrating that article fifty years later, we realise immediately that this is the *preservation*

of oral tradition by more modern means. We see therefore that the particular discourse of mathematics is not simply a proof coming forth to the reader; it is a narrative, with its own fictional universe, and fictional entities such as C^* -algebras, or probability densities, or rational varieties; entities inaccessible and invisible to the uninitiated. An article in other words is a modern version of an ancient Egyptian scroll, with its own hieroglyphs understood by some, with a particular originator, which is carried through time by the younger generations, who then, in their turn, get inspired to construct their own Egyptian scrolls. It has to be borne in mind, though, that an Egyptian scroll is important to some and not to everybody. Those interviewed were mathematicians who were somehow fascinated by a certain branch of mathematics and pursued their interests persistently up to the point of receiving later recognition in terms of tenure. Not all of their fellow students ended up professional researchers in mathematics. In one such case, the interviewee ended up working for the private sector after the dissolution of the Soviet Union: he left the community and went on in pursuit of different ancient scrolls.

The authors, therefore, of published articles in mathematics still have *ownership* of their articles, but no longer *possession*: the whole community now shares them and has property rights over them. And sharing an heirloom with the community detonates reciprocity of a special kind: “to give in return does not mean to give back, to repay; it means *to give in turn*” [174, p. 48, my emphasis]. One of the interviewees described his time as a postgraduate student in the Landau Institute on Theoretical Physics:

“I mean, it was very good. It was a very interesting place. Approximately around that time they were creating the theory of conformal invariants, the theory of two-dimensional fields: Polyakov, Zamolodchikov, Migdal, Belavin, Knizhnik. I mean all these people were there, and were, kind of, creating that theory. Just to watch it was amazing. We were doing our stuff, we were doing our research, which was quite interesting in itself. There were very few mathematicians there... and then I was accepted as a mathematician ... It was a small group of mathematicians but it

was very well integrated.”

To put the reader into perspective, Alexander Polyakov and Alexander Migdal were among the first in the 60s to publish a paper on what later came to be known as the *Higgs mechanism*, named after the the British mathematical physicist and Nobel prize laureate Peter Higgs. In such an exciting research environment,

“men who give more [gifts] than they have been given, or who give so much that they can never be repaid raise themselves above other men and are something like gods, or at least they strive to be” [174, p 30].

And an interesting aspect of gift giving is when it is conducted between people of different social position or rank, since sharing decreases the distance between the two parties, while at the same time indebtedness increases their social distance. In a conference in Lomonosov University, during the author’s fieldwork, the head of the organising committee was Albert Shiryaev, a major contributor to stochastic calculus and financial mathematics. After he delivered the introductory speech there was a break. As it would be expected, many participants during the break, went to greet him and talk a bit with him. But there was a rather unexpected incident for such a rather solemn event: one participant asked Shiryaev to take a picture with him. Now this would be expected to happen with a Hollywood star who enjoys a celebrity status, not with a modest mathematician. From the eagerness of the majority of participants to greet professor Shiryaev, to the rather extreme case of asking a picture with him, all point to an undeniable fact: celebrities do not exist only in Hollywood show business, but to any professional field, irrespective of the existing political regime this celebrity lives in. And only gods, or those close to gods enjoy celebrity status.

So when a postgraduate manages to publish an article he donates it not only to researchers of equal status, but to researchers of very high status, of a god-like status. But when a member of the community of god-like status publishes an article to whom, besides his or her peers, he donates his or her

gift? When a member is close in terms of status to the sacred spiritual world, who can be of higher status? Actually “giving [a gift] to a superior does not necessarily imply that the recipient is a human being” [174, p. 13]; it can be now a disembodied god, an immaterial one. But what kind of god can exist in a rational disciplinary enterprise such as that of mathematics? In fact, there is a candidate god in a disenchanted scientific world such as ours: *scientific truth*. Scientific truth, a concept created long before the rise of the communist states, is supposed to correspond to some kind of objective reality. If this “‘correspondence’ is supposed to be utterly independent of the ways in which we confirm the assertions we make [...] then the ‘correspondence’ is an *occult* one, and our supposed grasp of it is also *occult*” [370, p. 10, my emphasis]. Scientific truth as correspondence to an objective reality in any discipline “can only be the combined work of imagination, metaphor and authority” [207, p. 242]. Scientists imagine an external reality which cannot be observed by humans,¹⁰ they employ metaphor to formulate narrative explanations, and they rely on the authority of the book, the article, or the lecturer to clarify truth. What is scientific truth in biology, or chemistry, and how is it related to mathematical truth? Can the methods of reaching truth in mathematics be employed by physicists, or the methods of sociologists be employed by biologists? The answer is clearly no. An interview is not data collection in probability theory, and proving a theorem is not proper methodology in biology; at least as yet. At least until there is a new fusion of methodologies to be imagined, a new metaphor to be formulated, and a new book or author to render a new methodology scientific status. Scientific truth, in other words, is a character in the fictional universe of the scientific community, hidden by public view as such, unofficially raised to the status of a god, to whom the learned elders of science give away their own published truths. The “Sokal affair”, for example, while it was allegedly an attempt to expose the “lack” of scientific validity in the humanities (see [137]), in fact, it was an attempt to impose methods of one *scientific cult* to another one, and it ended up by exposing the *necessary illusion* that the scientific enterprise can be unified in its pursuit of a transcendental truth as its own grand narrative; and exposing

¹⁰Gravity, for instance, has never been directly observed; only its effects can be observed and measured.

illusions is necessarily a controversial event.

Any scientist, of course, can lay claim as to the personal motivations of why joining a particular field, and that will be indeed valid. Curiosity was the most main reason in the interviews, and personal interest. Some were involved in mathematical activities quite early in their lives, or at least, they traced an interest in mathematics quite early in their lives during the interview. Others added the fact that there was more freedom in mathematics research during the Soviet period, when compared, for example, to other disciplines, such as physics or chemistry. In one such case, for example, when the interviewee had to choose between mathematics and physics, he chose mathematics, because, as he had been told, studying physics involved laboratories as well, besides theory, and laboratory equipment involved many more approvals by higher ranking officials, than those needed by a mathematics department. Other interviewees mentioned personal ambition and eagerness not only to survive in a communist society, but to thrive, as well: being a university professor in the Soviet Union commanded quite a high respect. People in general can give as many reasons for their personal motivation, as the number of those asked. But when a god of science comes in, when a divine entity enters the scene of science, it enters “idealized, transmuted into the common good, into a sacred principle which brooks no argument, no opposition, which can only be the object of unanimous consent” [174, p. 174]. Let us not forget that Marxism-Leninism considered itself to be scientific. Moreover, any university graduate, in order to be accepted as a doctoral candidate in a higher institution, had to be examined and succeed in three areas: his or her own disciplinary field, a foreign language considered appropriate for his or her field, and scientific communism. There were even textbooks on scientific communism, especially for students of tertiary education (see for example [146]). It was, therefore, not difficult in the USSR to establish a new institution of scientific standing, once the approval of the corresponding regulatory bodies was acquired. But, in any case, raising a building of baroque proportions such that of the Main Building of the Lomonosov University, and hosting a large group of people under its roof, can be done only when a divine entity, such as that of scientific truth, comes down to earth from his

residing place in heaven. Such a divine entity unifies a large number of people of various persuasions and motivations under its auspices; and, moreover, it provides its people with a legitimate pretence and an encompassing local grand narrative for becoming organised around a gift economy, such as that in the Soviet scientific institutions.

Gift economies in general have attracted the attention of cultural anthropologists for many decades. Mauss was the first to consider gift transactions among indigenous people [312, 313] and connect it to specific social action, or *prestation* in his own terminology. It is interesting to note that the first translation into English of the *The Gift* the translator found “no convenient English word to translate the French *prestation*” [312, p.xi], while in the most recent English translation the editor notes that the French term has “been referred to in the translation for brevity’s sake, as ‘total services’,” [313, p.vii]. According to the Langenscheidt’s Standard French Dictionary, *prestation* can be translated as (money) lending, (insurance) benefits, allowances, or services (performed). What Mauss was trying to convey with this lexical relic of feudal France was a meaning of “a service performed out of obligation, something akin to ‘community service’ as an alternative to imprisonment” [204, p. 50]. So what Mauss actually accomplished was to point out, quite eloquently one could say, the deeper connection between artefact materiality, social action, and individual, or group, identity of *Homo sapiens*.¹¹ The difference between a gift economy and a modern capitalist commodity economy “is that the latter is created by the exchange of alienable objects between transactors who are in a state of reciprocal independence,” whereas the former “is created by an exchange of inalienable objects between people in a state of reciprocal dependence” [188, p. 640] (see also [189]). While commodity transactions, such as oil or gold, are based on a mutual independence, and can be conducted even among strangers, gift transactions, such as talismans or wedding presents, are based on a mutual dependence and cannot easily be conducted among strangers. Moreover, while a gift transaction leads to a *qualitative* relationship, a commodity transaction establishes a *quantitative* relationship between the transacting parties. Commodities have a financial

¹¹That is, of the modern, and last, descendant species of the *Homo* genus.

price, but gifts have an emotional price. Scientific articles can be considered, as well, commodities, in this quantitative sense, since a researcher can always cite the number of articles published in scientific journals, or the number of articles published by other researchers based on his own articles. So a published article can be considered, up to a point, that it stands in between a gift and a commodity.

A commodity, due to its *fungibility*, can easily be alienated from its owner temporarily, in the case, for instance, of a deposit: a barrel of oil can be given to a safe-keeper for a while, and then retrieved back the same *in quantity and quality*; there is no need to ask back for *the very same* barrel of oil that was given in the first place. Fungibility, on the other hand, is not a property of a scientific article: an article cannot be given back in half, or an author cannot publish a “quarter of an article; but he can publish parts. Moreover, a commodity is fungible, that is, it can be subdivided by trading parties into smaller quantities, because the exchanging agents are interested more in its *marginal utility*, rather than its *total utility*. The utility function is a concept used in neoclassical economics, and has been defined as a multidimensional (mathematical) “function that specifies the utility (well-being) of a consumer for all combinations of goods consumed (and sometimes other considerations)” [114, p. 287]. In mathematical terms, a utility function is assumed to be *differentiable*, that is, its partial derivatives exist, although this property of differentiability, in general, “is almost always imposed but has not been translated [adequately] into behavioral terms” [342] (see also [116] for a mathematically more rigorous exposition). However, if one needed to employ a non-mathematical narrative, rather than the geometrical one from differential geometry and measure theory, to “explain”,¹² for example, to an undergraduate audience,¹³ then *utility* could be defined as this “common something that enables two heterogeneous things to be compared and valued so that a *choice* can be made” [190, p.18, my emphasis]. In this way we can deduce that “the marginal utility of diamonds can be very high (because diamonds are very scarce) relative to the marginal utility of water (because

¹²Or rather to legitimate its definition.

¹³That is to university students, as well as, to non specialists.

water is very abundant)” [159, p. 39].¹⁴ The economic utility approach can partly account for the fact that the number of publications on stochastic analysis today has proliferated in exponential function, while those on ergodic theory has remained rather stable. Both fields descended from probability theory; on both fields there have been substantial contributions by the Soviet mathematicians; but, while stochastic analysis is heavily applied on financial research and practice, that is, a field in high demand today, ergodic theory is used in statistical (quantum) physics, a field with demand limited mainly to academic research institutions.

By the late seventies and early eighties the mathematics departments were becoming in general very popular among school pupils: they offered a career prospect with a secure future insulated from party interventions. In the words of one interviewee “many students wanted to *be* mathematicians, rather than *work* as mathematicians,” meaning that they aspired for the security resulting from tenure, rather than for the research presupposed by tenure. And considering the fact that in order to proceed to postgraduate studies an undergraduate was expected to have published within the five years of his or her studies at least 2 scientific articles, it could indeed be claimed that scientific articles were important economic goods, with a high slope in their marginal utility: they influenced substantially the personal choices of the future researcher with respect to their projected marginal utility. In gift transactions, as we saw, “objects are personified,” while in a commodity economy “persons are objectified” [348, p. 233]. It should be borne in mind, at this point, that human labour, although it is not something material such as oil or gold, and cannot be measured in grammes, litres, or seconds, in our modern “capitalist” Western society it has become a commodity and can be sold, rented, or acquired, as a normal material commodity. Whether a scientific article is, or can be considered to be, an economic good, a commodity, or a gift, it is because what lies under all these assumptions is a deeply Western concept: personal property, whether this person is a concrete, biological *Homo sapiens*, or an imaginary, legal person managed through human

¹⁴The marginal utility of the diamond is the partial derivative of the total (or general) utility function with respect to the diamond dimension.

trustees. Soviet communism, in fact, did not change the Western concept and experience of property, which dates back at least to the ancient Greek and Roman antiquity, if not earlier. Communism changed only the biological trustees managing property, according to how property was distributed in its own social and political context. From anthropological research conducted in non Western societies, such as Papua New Guinea, it has emerged that the idea of property in these societies and its value is related with “the person’s *own* appropriation of his or her activity that gives it value, in so far as the person is a microcosm of the social process by which exogenous appropriation by others, by the system, also gives it value” [424, p.142–143, emphasis in the original]. The whole first chapter of this thesis was actually a theoretical anticipation of an inescapable fact: materiality is *hardwired* on the human brain, and gift transactions are actually its manifestation upon social action, in the same way as hunger is hardwired on the human brain, and regional cuisines are the manifestation of hunger upon social action. Materiality, whose one manifestation among many are the regional cuisines, in fact, seems to emerge as a biological instinct, rather than just a theoretical conceptual tool. But we will come back later on in the text on this theme.

4.3 Value and its Spacetime of Content

On the 26th of April 1965 the American mathematician Paul Halmos arrived in Moscow. He was there as a result of “recent exchange agreement between the Academies [of science] of the USA and the USSR,” [the purpose of which] “was that each country was to send to the other 20 prominent scientists some time during the years 1964–1965, ‘at least half of whom shall be members of the respective Academies, for a period of up to one month each, to deliver lectures, conduct seminars, and to study scientific research on various problems of science’ ” [202, p.290]. Dmitrii Anosov, a mathematician who would make major contributions in dynamical systems later on, was assigned as Halmos’s translator and guide. Halmos, gave some lectures in Moscow, and then in Leningrad, met with various other mathematicians, with Kolmogorov, took part in Yakov Sinai’s seminar, and did also some tourism [17]. Anosov would

later in that year receive his Soviet degree of Doctor of Sciences [16]. Halmos was a well known mathematician both in the USA as well as in the USSR. He had published influential textbooks in measure theory and on Hilbert spaces, which were translated into Russian within less than a year. Many interviewees mentioned his books, especially when asked specifically questions on how they acquired knowledge of infinite-dimensional spaces. Halmos in his autobiography mentions the touristic attractions he visited in a rather vivid narrative, along with his talks and discussions with Soviet mathematicians; he was keeping a daily diary all over his life, and probably this helped him as to the vividness of his “exploits” there. Tourism and travelling are, of course, personal events, not scientific ones, and Anosov’s report on Halmos’s stay ([17]) was not as vivid as Halmos describes it in his memoirs. One of the interviewees in the Ukraine, after the end of the interview, he pointed to a big world atlas hanging on the wall which had pin flags attached on it and he explained, as if bragging, that each pin flag represented a country or a place he had been for a conference, or for some other reason under the capacity of a researcher mathematician. I have to admit that I counted at least thirty-forty such pin flags scattered all over the atlas. He seemed to be bragging about these, but most probably this was faked bragging, because there was a junior researcher present, translating from English into Russian, and vice versa, during the interview. Most probably this was one of his ways to motivate the young mathematician as to the potential side benefits of research in mathematics, one of which could be travelling as well as tourism. In the archives of the Steklov Mathematical Institute, on the contrary, we do not find such reports of tourism and travelling, not just because these reports are written in bureaucratic language, but because these activities are considered as unrelated to the business and conduct of scientific research.

Instead of talking about travelling as coincidental and “unrelated to the business and conduct of scientific research,” we could actually reformulate this last assertion of the previous paragraph in different terms: travelling and/or tourism do not assist in publishing articles, that is, as activities themselves do not improve the social-scientific standing of the mathematician, or rather, they do not *bring in value* to scientific activities. What brings in value to

science can be a discussion with other researchers, attending a conference, giving a lecture and answering questions, and so on. And again we bump into the question of value once again, since in a gift economy, besides gifts themselves there is the act of gift-giving, possible discussions before and after gift-giving, creating acquaintanceship with other regular gift-givers, or gift-constructors, or even planning a course of a sequence of gift-givings. The scientific article itself, in other words, is not the only source of value, but many activities leading to that, or even simply reading one, can be valuable. So the next step in identifying the gift economy of scientific activity in the Soviet Union is finding a way to define economic value, but that kind of value that is intrinsic to science. The first thing to notice is that there can be as many definitions of value as there are people alive: everybody will have their own definitions, whether economists, or sociologists, or, in general, any member of the general public. An ambitious politician would find value in improving her oratory skills, and a aspirant mafia leader would find value in killing certain competitors; a professional dancer would find value in learning some new dancing moves, and a computer cracker would find value in writing a script to hack an ATM machine. It would be also scientifically prudent not to give our own arrogant definition of what is valuable, a common research custom in many scientific circles. It would be wiser, as social scientists proper, to “let the people speak,” to let, in other words, social action define itself, or rather, to let the specific group in question to define what is valuable to it, and what is not. A gift economy, after all, is not created only by objects, but by relations between its human agents, by the circulation of gifts, by the action plans and the desires of human agents, by the creation of new material and social relations and so on. So “[r]ather than having to choose between the desirability of objects and the importance of human relations, one can now see both as refractions of the same thing” [180, p. 45], that is, both as refractions of economic value. And at this point, anthropological research on economic value, especially research on stateless societies, has some very interesting things to contribute. A scientific community, in the end of the day, is a stateless society.

Let us begin with a fictional university student in Moscow during a normal

day of his studies. He would wake up in the morning, after breakfast would catch the bus, or the tram, and after some time he would get off at the Lomonosov university's Main Building and stay inside most of the day, until he went back home. This fictional student, in other words, during one of his normal days, he moves from one enclosed space into the next one. This scenario would not be very plausible in a Mediterranean country such as Greece: students there can go out on the parks as well, or to a coffee shop, and study, for example. In Moscow, on the contrary, especially during winter time, the temperatures are so low, minus 20 degrees Celsius on average, that it is almost impossible to stay outside of a building for very long. When in one day of February, the author made the mistake to buy a hot chiburekki¹⁵ from a street bakery, with the intention of eating it on his way to the archives, he felt this aspect of living in Moscow quite strongly: the hot chiburekki within thirty seconds was already cold, and his naked hand holding the chiburekki was already frozen and after a minute of walking he had to finish eating and put his exposed hand back to his pockets; after a whole minute of holding the chiburekki he could hardly move his fingers because of the cold. It was the same situation during fieldwork in Kiev. Even smoking outside is also very difficult, but at the Soviet times smoking indoors was something both legal and normal. Most of the social life in Moscow, in other words, at the end of autumn until the beginning of spring happens within enclosed spaces, and people do not stay out for very long, unless there is a reason for that, as in the case, for example, of police patrols, or militia patrols in the Soviet times. This is one of the reasons that in Kiev, for example, there are a lot of underground markets and shopping malls. During autumn, as well as during spring, there is also a lot of raining, at least in Moscow, which keeps, as well, people indoors. Summers in Moscow, as well as in Kiev, are very close to Mediterranean ones: lots of sunshine and high temperatures, sometimes even 30 degrees Celsius. Our fictional student, in other words, would stay most of the time of the year during his studies somewhere indoors. A similar kind of life would live a postgraduate, or a researcher. In other words, the scientific community, in Moscow, as well as in many other major cities, such as Kiev, Leningrad, or Novosibirsk, was, and still is, very close to that of an

¹⁵It is a traditional Crimean Tatar type of a pie filled with minced meat.

island community in Papua New Guinea, or to the communities of a complex of scientific research stations in the Antarctica. Everyday mobility in this kind of community is, in other words, similar to moving from one island to another.

The archetypical peripatetic school of ancient Greece, in other words, in which the student philosophers walked side by side with the teacher philosopher, could not have existed in Russia under these weather conditions. But academic buildings, such as the Main Building of the Lomonosov University served another purpose, apart from protection from the weather: the industrialisation of higher and tertiary education. After the victory of the Bolsheviks in Russia, Vladimir Lenin and many others saw it as vital to eradicate the high percentages of illiteracy inherited from the Imperial Russia: they could not survive otherwise in a hostile “capitalist” world. A special committee was formed, People’s Commisariat for Enlightenment, *Narkompros*, whose sole purpose was the eradication of illiteracy. This campaign was named *likbez*, literally translated as “liquidation of illiteracy (for more see [96]). After the second world war Narkompros was renamed into Ministry of Education. During the 50s, when most of the older generations of the October revolution had died, the literacy rate reached almost 100%. Then a second wave of expanding the “higher education literacy” system to tertiary education took place, especially during the Stalinist period, that is from the 1930s to the 1950s. This wave swept over mathematics research, as well: Steklov Mathematical Institute was created in 1934; St. Petersburg Department of V. the Steklov Institute of Mathematics split from its mother institution in 1940; the Institute of Mathematics and Mechanics in Ekaterinburg was founded in 1956; the Keldysh Institute of Applied Mathematics in Moscow in 1966; the Sobolev Institute of Mathematics in Novosibirsk was created in 1957; the Landau Institute for Theoretical Physics outside of Moscow in 1965; to name just a few, apart from mathematical institutes all over the former Soviet republics, as well as institutes employing a great number of mathematicians to conduct research such as the Kharkevich Institute for Information Transmission Problems, in Moscow founded in 1961. To give a comparative view of the size of these institutes, as one interviewee

mentioned, who was a postgraduate student in the 1970s:

[The] Landau Institute was [an] extraordinary organisation. It was the strongest institute on theoretical physics in the world at the time. [...] So, by Western standards it was a rather big institution, which had maybe eighty plus researchers, by Soviet standards it was [a] negligibly small institution. The institute was eighty people, it was just nothing.

The reader should also bear in mind that so far postgraduate institutes only in mathematics have been mentioned, not full-blown universities, with their own additional postgraduates and researchers on mathematics. The most well-known equivalent such postgraduate institute in the West, is the Institute of Advanced Study in Princeton, where Albert Einstein worked after the second world war until for the rest of his life; but the number of its employees is much less than those of the USSR, and moreover, it employs researchers from all over the scientific disciplines, not only from mathematics.

One need that led to industrialisation of education, in general, was the continuous supply of academic researchers. But besides buildings to house future researchers, there were some other technologies employed, basically borrowed, which usually go unnoticed, but still are vital. Due to the great number of students, the expansion of tertiary education leads to increased complexity problems. In one of the interviews, for example, the interviewee, in order to show some part of a mathematical proof, invited the author to the table she was sitting, picked up a pen and begun writing on a sheet of paper which she had already placed on the table. This could be repeated up to a certain number of attendants. If the number of those in her audience had risen to a hundred, for example, using a sheet of paper placed on a table, the proof could not have been demonstrated to the audience: there would be too many people, or rather, too many human bodies, to fit into such a small place. The lecture theatre, along with the use of the blackboard solved this mundane, but fundamental problem (see Fig.4.1). The lecture theatre is actually a special case of the Roman amphitheatre, which in its turn descended from the ancient Greek theatre. Amphitheatre, which originates from the ancient

Greek language, means a theatre from both sides: in an ancient Greek theatre the spectators were sitting in a semicircular fashion in each row, and each succeeding row was raised higher from the previous row, as the spectator was sitting in longer distance from the centre of theatrical activity. The architectural success of the theatre was employed by the Roman emperors to build “double theatres,” the most famous of which was the Colosseum, which “experts believe it could have held 50,000 spectators” [75, p. 1075]. The lecture theatre solved the problem of talking to a large audience, as well as, the problem of establishing face-to-face communication of each student with the lecturer during the lecture: any student can ask the lecturer something, and the lecturer can answer directly by looking at the student while talking to him. The height of the Main Building of the Lomonosov University also serves to house more people in less space: it is comprised of thirty-two floors,¹⁶ thus scaling up the operations of the university, and handling at the same time problems of increased complexity.

Any building has to provide a life-support system for its temporary or permanent inhabitants. Its main purpose is to protect from natural phenomena which can be threatening to the human health, such as cold and rain. In the Soviet case protection from very low subzero temperatures, is an additional case in point. Air usually is not such a big problem, unless it is heavily polluted, a rather plausible scenario today.¹⁷ A university building, therefore, has to provide to its inhabitants with a life-support ecosystem, its own artificial ecosystem, at least for as long as it houses its inhabitants. Moreover, the inhabitants within the building, form their own temporary small society of everyday life. In the Main Building of the Lomonosov University, there are students in classes, and students out on the corridors; there are secretaries talking with professors, or with students; security guards checking the docu-

¹⁶That is 31 floors in the British scheme of floor numbering. In Russia the first floor is the ground floor, as in the American scheme of floor numbering.

¹⁷In China, for example, there are many cities suffering from heavy smog, one of which “was hit by 129 days of ‘unhealthy air’ or worse – the threshold at which pollution is considered at emergency levels” [447]; one could easily imagine that in cities such as these, the university buildings would keep their windows closed all the time and would be equipped with special air-conditioning systems.



Figure 4.1: Typical lecture theatre in the School of Mechanics and Mathematics, Lomonosov University. One can easily see the blackboard on the bottom, and the (modern) projector screen.

ments of those entering. Steklov Institute building, on the contrary seems a rather more dull place: the main reason is that there are no undergraduates there, although there is more social activity on the self-service restaurant (*stolovaya*) on the ground floor. There are also lifts in the Main Building, which are used on a constant basis. In lifts, as well as in buses and trams for example, unacquainted strangers may come into interaction: if one person is waiting on the fifth floor, wants to get to the fifteenth, and the door of the lift opens, he asks whether the lift goes up or down, in order to decide as to whether he will board on the lift or not; either he is acquainted with the person he asks or not, the other person may answer, or otherwise his silence would be considered as impolite. In Russia, in particular, the “thou” pronoun (singular grammatical number) is used in a familiar setting, and the “you” pronoun (plural grammatical number) is used either in a setting of people more than one, or as a pronoun of politeness; this distinction has been lost in modern English. Returning to the previous scenario with the lift, we would say that this brief information and politeness social exchange between two otherwise strangers to each other, may happen because of the lift itself: they come into a brief moment of acquaintanceship under the capacity of “fellow users of a public place” [175, p. 7, n. 5]. This scenario is actually more widespread in science: if a scientist has published an article that is relevant to some other scientist’s published research, during a conference, for example, the latter could take the initiative to become acquainted with the first one with the purpose of further discussion: they have become both of them “fellow users of a public artefact” [i.e. a scientific article], whether they were previously personally acquainted with each other or not.

One of the students of Vladimir Arnold, a major figure in mathematical physics, described his days at the Lomonosov University as an undergraduate as follows:

I spent the first half of the day at the university. Of course, as far as I remember, the courses started at nine o’clock, or something like that; then the courses lasted until two or three o’clock in the afternoon; then we had a lot of special courses, a lot of special seminars, and so on. So if one particular day I had these

additional courses and seminars I was at those courses; otherwise I went home, and I worked at home; and then, of course, even in the days when we had our special courses, I went home, and I worked at home. And actually this was from Monday to Saturday. Saturday was a normal working day. And only Sunday was a day off. But all the time apart [from] my studies in the university, I worked at home, so I did my home work, exercises. I worked at the scientific problems Arnold gave me, and so on. So actually all the time from early morning to late evening, all the time, 7 days per week were devoted to studies, to studying and so on. [...] This was typical for most good students.

Very few of the interviewees described their first year as a year without much studying and more like socialising only for the purposes of leisure, but then from the second year on their everyday undergraduate life was similar to that described above. Postgraduates and researchers conducted a life with a similar pattern, but with less pressure like that of the undergraduates. Moreover, since only the heads of the departments had an office, most of the postgraduates met with their supervisors at the supervisors' home. The whole mathematical community of the university, in other words, was a virtual island community, rather separate from other communities: its people were enclosed most of the time, and its members met with one another most of the time only with other members of the same community. Let us not forget that in that time there were no social electronic media, such as Facebook, or Twitter, to enhance social interaction among large parts of the population, as it has been happening for the last 5 years of today. In fact any professional community can be considered as a virtual island community, depending on how the everyday life of its members is organised, and on to what extent restrictions of membership exist.

If therefore we consider for a moment the mathematical university communities over the whole geographical area of the former USSR as virtual island communities, all of them organised around a gift economy of published articles, then what begins to emerge is “[a]n intricate time-space-person system covering hundreds of miles and several decades, linking many hundreds of

people in respect to thousands of strictly individual objects [i.e. scientific articles]” [363, p.12–13]. The members with this island world “can make connections not only with ‘consociates,’ but also with more distant contemporaries with whom they have few or no face-to-face relationships” [334, p.6]. If one, additionally, considers the Gelfand-Naimark theorem, which was previously mentioned, then it can be said that connections can be made even posthumously across the physical time continuum; the whole modern research movement on *noncommutative geometry*, along with its implications on *quantum field theory*, is practically based on this single theorem (see [211, 225]). The case with the impact of the Gelfand-Naimark theorem entails a certain *mode of institutional spacetime* [334, p.9–11]: Gelfand and Naimark themselves, those mathematicians before them whose mathematical results Gelfand and Naimark used; those after them who used Gelfand and Naimark’s theorem. The buildings and the artefacts, such as pencils and papers, being used by each one involved around this theorem. Due to the gift economy of scientific articles people who are not personally acquainted can be connected as well. Another mode of institutional spacetime are conferences, as we saw earlier with Halmos. The Main Building of the Lomonosov University, due to its endurance over spacetime, whose materiality provides the *spatiotemporal stability* of the institution. A small child, and indeed any person, who sees for the first time the Main Building will remember the Lomonosov University *as that building*, although the building itself is not the institution. Predecessors, contemporaries and successors are continuously connected over a geo-historical spacetime: one interviewee mentioned Halmos’ book *A Hilbert Space Problem Book* although he had never personally met Halmos; many interviewees narrated stories about people they were personally acquainted with, such as supervisors, or colleagues, or stories about people they heard from third parties. There is a multitude of published articles “celebrating the 60th, 70th, or 80th anniversary” of this or that professor written by former students; these published articles are, in fact, monuments of institutional memory, rather than scientific articles proper. Another interviewee mentioned that “he knew how the system worked” and he planned his career accordingly: he was talking about a *projected institutional spacetime*, what today is commonly referred to as a “career.” A mode of institutional

spacetime, quite important in other parts of society, is *bodily spacetime* [334, p. 16–18]: in the case of the mathematical community of modern Russia, and from many photographs and a few Soviet documentaries on science that the author saw and watched, there was, and still is, no particular dress code observed apart from some very senior members of the community, who wear on a regular basis costumes, as an indicator of the formality of their official position, rather than as a personal choice. The dress code is rather plain, the process of dressing itself, as in any modern Western society, is conducted in a private space away from the institution’s building, and its purpose is to go unnoticed, rather than attract attention.¹⁸

The spacetime of social practices, that is, patterns of social action repeated on a regular basis, such as attendance to classes, solving problems, or going to examinations, and so on, is an important mode of institutional spacetime. Ivan Vinogradov, a very important Soviet mathematician with substantial contributions in analytic number theory, was the head of the Steklov Institute of Mathematics in Moscow, during 1934–1941, and then from 1944 until his death in 1983. In his report for the year 1965 [15], he mentions the following as his activities during the year in the first paragraph of the report: director of the institute and head of its department of number theory; chair of the national committee of Soviet mathematicians; head editor of the journal “Izvestia Academy of Sciences, Series on Mathematics”; member of the organisation committee and chair of its section for the coming international conference of mathematics to be held in Moscow in 1966. He mentions the positions, and not the activities springing from them, which means that he already assumes that the readers of the report, who are going to be the members of personnel administration of the presidium of the Academy of Sciences,¹⁹ already know what these activities specifically entail. The position of the head of a mathematics institute can be said to be occupied by a member of staff of extended experience, “extended experience” meaning,

¹⁸This can be contrasted to the body decoration during certain rituals in some native stateless societies, or to that in modern fitness clubs, which provide public outlets of private space to change clothes.

¹⁹Today we would talk about the human resources department of the Academy of Sciences.

among others, that he or she has identified the important patterns of social practice and he or she can become an accepted guardian (proper) and a visible imitation example of the social practices mode of institutional spacetime. A guardian of institutional practices, in other words, among other things, keeps the institution's projected spacetime stable and predictable. In the second paragraph of Vinogradov's report he refers to his own research activities in mathematics, and in analytic number theory specifically: he may be a guardian of institutional practices, but he never stops to be a member of a research community, still giving away his own gifts to the community, i.e. publishing his own articles. Institutional practices, however,

“do not simply go *in* or *through* time and space, but ...they form (structure) and constitute (create) the spacetime manifold in which they ‘go on.’ Actors must ‘make’ this manifold, thus concretely producing their own spacetime” [333, p. 280, emphasis in the original].

During their undergraduate years every student had an advisor, who at the same time was a supervisor. There were many seminars offered to the undergraduates, and usually these advisors were the organisers of these seminars. These seminars were optional as individual courses, but it was compulsory to choose a number of them. In addition, these courses were deeply related to the course organiser's research at the time. One such seminar in the Lomonosov University, for example, was Yakov Sinai's course on ergodic theory and dynamical systems, a pioneer in statistical mechanics and chaos theory; Sinai received the Abel prize in mathematics for lifetime achievement in 2014. Another seminar was Vladimir Arnold's, which later became a well known book in mathematical physics [18]; although Arnold's book is intended for undergraduates, or rather Soviet undergraduates, its material is very advanced for the undergraduates of American and British universities. These seminars attracted many students, and the advisors gave the students problems to solve, which were actually unsolved problems at the time. This pattern of undergraduate supervision was quite common in most of the interviews: the most frequent phrase in the interviews was “he [or

she] gave this [or that] problem to solve.” A very basic social structure in the social organisation of scientific research is supervision. In the case of mathematics in the Soviet Union this involved a certain component of its gift economy: each advisor gave a problem to a student to solve; then the student reciprocated with a solution. Each problem was usually something unsolved at the time, so it was, in fact, a future potential article when it was given to the student. The whole of the gift economy of the Soviet university was fundamentally organised around the social structure of supervision. In this special kind of gift transaction the first thing to note is that supervisory “activities which produce the [supervising, or a supervised] person also produce the sociocultural system and *vice versa*” [144, p.3]: the current supervising, and a past supervised, party transmits the particular scientific culture to the current supervised, and a future supervising, party. In this way supervision, at least in the Soviet case, created a cyclical process which connected the past of the mathematical institution with its future. Quite often the students went to the professor’s house, because there was limited space provided for office purposes; only the heads of the departments and faculties were provided with office space. Many of the interviewees declared their supervisors, either those during their undergraduate, or postgraduate years, as their friends, and many of them are still in touch, quite often. This social structure, or rather process, of supervision, in order to be considered as supervision is necessarily of fixed directionality in terms of a gift transaction: the supervising party gave an unsolved problem, and the supervised one, reciprocated by giving it back unsolved. This was happening, of course, due to the *functional differentiation* within the institution. Supervision, in other words, “include[s] the production of the objects exchanged and [of] the producers themselves” [144, p.3]; it was a fundamental component of the institutional spacetime of a Soviet university. If one considers the examination diets, assignment submissions, as well as their marking, one can easily realise that the gift economy of the university institution is much more widespread and more heavily institutionalized than previously thought.

We see, therefore, the genesis of structural loops of everyday social practice on the microscale: a certain activity is considered a necessary and proper,

therefore it has to be performed, and because of its performance “on the job” it is considered as proper and therefore it feeds back on its necessity. The actor creates his or her own institutional spacetime which he or she inhabits at the same time. In the case of activities which are not described by some official textbooks, or prescribed by instruction, and the actor has been “left to his or her own devices” as to a non-prescribed instance of social action, something quite similar emerges. When one interviewee was asked what kind of advice he would give to a student in order for the student to develop his mathematical imagination his answer was:

So, you know , this was not created by me, I just follow my teachers. So what they did, they just suggested a lot of problems, quite different. Some of them very simple, some of them were just open problems , some of them very famous problems, like Hilbert problems, during the century. You know, of very different level. And the point is that it is up to the student. The student has to choose [for/by] himself. So basically, you just, maybe my feeling is that he teacher has to show different ways, different possibilities, different opportunities.

The interviewee in other words upon being questioned was in position to *identify by himself* the particular type of social action connecting him to an imaginary successor, to *reflect* on it without being pointed to it by an external authority, *ascribe* the origin of *his own* particular choices to his imaginary, at the moment of the interview, predecessors, and consider his assertions as valid. None of the interviewees reported any official instruction during their mathematical career in the Soviet Union, as to how supervision should be conducted, or as to what type of supervision would be considered as more appropriate; all the interviewees, therefore, had inferred that themselves, and then considered it valid by simply performing it, that is, as a self-fulfilling prophecy. They all were “self-taught ethnographers” for their own purposes without the usual notes of a “proper,” or rather a “professional,” ethnographer.²⁰ Classroom instruction, for instance, is not the best type of

²⁰That is, a member of the virtual island community of the social sciences.

instruction, in general, but “it works,” or “that’s the way it has always been,” or “everybody does it.”

We can now proceed to a preliminary consideration of economic value as *extendable* institutional spacetime, that is, “the capacity to develop spatiotemporal relations that go beyond the self, or that expand dimensions of the spatiotemporal control of an actor” [334, p. 11]. Since economic value is always related to a projected future, there is always some design and planning involved. One of the interviewees, when asked about his reasons of choosing mathematics over other subjects, while at school, he answered as follows:

I had some broad interest in sciences, in general [...] But physics and mathematics were my favourite subjects. And including astronomy. And at some moment I was doing well in both Olympiads, in math and in physics. [...] And then at some moment I had to decide. And it was late sixties, pretty deep in the Soviet Union, and I realised I wanted more freedom ... including my career too. I also realised that if I choose physics I will be very much dependent on the state system. Because you cannot do big physics without big money, and big experiments, and equipment. [...] So between two equally attractive alternatives, I decided that mathematics will give me more freedom and doing what I want.

We can clearly see that what is commonly called today “a career” is actually one projected institutional spacetime over others. What we also saw in the previous section as marginal utility is actually a projected spacetime for one more reason: a school pupil can think about attractive potential futures and choose one over the others through *his own* experiences, and through his *his own* discussions with with other people, such as parents or relatives, and not by calculating the partial derivative of some marginal utility function; at least not in the case of the above mathematician, and of all the other interviewees as well. In this particular case we see an actor deciding over which virtual island community to join by projecting various branches of the institutional spacetime of his experiences at school and especially of those in the mathematical Olympiads. By joining, later on, a community, he will start

participating in the construction of and constructing his participation in the community's spacetime. By thinking about the extension of his spacetime, he paves an alternative future spacetime of the virtual island community. In a similar fashion, we can see how recruiting postgraduates, discussing with other researchers, or going to conferences, has institutional economic value, because these activities lead to a projected extension of the institutional spacetime. What is important at this point, therefore, to notice, is that, in fact, "the [scientific] community creates itself as the *agent of its own value*" [334, p. 20].

One particular case narrated in an interview is worth elaborating at this point, because it is a nice instance of taking an initiative and invest in this particular gift economy. The interviewee was asked whether he could mention any example of a bright or talented students. His answer was as follows:

At the time, probably even now, it was prestigious to do things like algebraic geometry, algebraic number theory, and many people were going to this direction, bright people. [...] I think it was '89, when for the first time entry examinations in the Moscow State University [i.e. Lomonosov University] was not related to any national issue, Jewish, non Jewish. Nothing. It was just honest examination. And so they got a very strong cohort of students. And I told [my colleague], "if we don't do anything, these algebraists will grab the whole pack and we will get nothing from it; so let's have a seminar, a seminar on dynamical systems and ergodic theory for first year students; first year, first semester." And we got a group like 10-15 guys coming, and one of them was Dima Dolgopyat. He was very very young. [...] And the plan which we developed worked perfectly, because he stayed in dynamical systems, later became student of Sinai in Princeton, a PhD student. And he's now one of the most interesting mathematicians in the area, and my close friend. [...] I like this example because it was designed to "fish" somebody very good, and we managed to "hook" somebody very good for dynamical systems.

In this particular case, we have an interesting example of an initiative that looks like investment behaviour: employing the resources of the institution to produce more value, that is, a projected extended institutional spacetime. The particular interviewee, first of all, had faith in the examination system, that is, he had *trust* in the new institutional spacetime. So he decided to invest his own, as well as his colleague's labour, in certain activities of value. The value of one's own labour, as long as labour is considered institutionally valuable, both in general, and teaching in particular, relies basically on whether an article will be produced in the in of the projected process. However talented a mathematician is, in the end of the day, his or her value as a mathematician becomes *visible* only through the publication of an article. In the case of the economic value of a supervisor, this could become visible through a potential publication of one of his or her students. A future article, in other words, seems actually to be the *only* end result of economic value in the mathematics research gift economy. With applied mathematics, for example, the benefit is not to mathematics itself, but to other fields that mathematics it applied on. In mathematics it is valuable to produce articles in the form of axiom-proposition-proof-theorem-proof-corollary. A researcher in material science would be interested in implementing ergodic theory and dynamical systems in his laboratory experiments, even in theorising about them; a mathematician in the same field, that is, ergodic theory and dynamical systems, on the other hand, would be interested in inventing mathematical entities, such as, probability measures, ergodic transformations, or compact manifolds, in order to formulate axioms, then formulate propositions and prove them, then formulate theorems and prove them, then formulate corollaries, and so on. Gravity is of interest to a physicist, not to a mathematician. The geometry of gravity, or rather the attempt to formulate and develop a geometrical model of gravity, on the contrary, is of interest to a mathematician. The physicist is interested in a geometrical model of gravity as far as this model *is* the reality, rather than an attempt to describe it; in a laboratory experiment the mathematical model is taken for granted, rather than developed on the spot.

Returning now to the economic value of mathematical articles, it seems that

the whole edifice of mathematics is organised around publications of new results. The whole of mathematical community is mobilised by theorems, axioms, corollaries, while at the same time these seem arcane to the lay public. In the end of the day the economic value of money seems to share a common function: mobilising social groups towards organisation. But this mobilisation is not circumstantial: gifts have to be exchanged first, in order to enter circulation. When an article is published this is a moment of exchange, that is, “the point at which the latent value created in production processes and embedded in the products is transformed into publicly recognized forms of value” [144, p. 8]. The same happens when a consumer takes out of her pocket a pack of banknotes to pay a product at the cash desk she has just selected: tending the hand to the cashier while holding the banknotes she displays her financial economic value of the spot. The definition of economic value as extendable institutional spacetime is a definition of subjective value, that is, of value as the actor experiences it, and it is closer to the social imaginary of the institution, which is visible mainly to its members. The articles, on the contrary have both an intersubjective and an objective value: intersubjective because they have rendered subjective value visible to everybody beyond the particular institution, and objective because they are made of material, either paper, or, in today’s scientific articles, computer screens. The scientific article, in other words is the meeting point between materiality, society, and consciousness, and as such it can circulate all over the world, and not just in the former USSR. An article, besides its content, which is visible to a specially trained subcommunity, is, in fact, a *token of value*, besides being an object of value. And as a token of value it can certainly now mobilise social groups which are known under the specialised name of “scientific community.” Another aspect of the scientific article, therefore, seems to arise: the scientific article, as a token of value, is the material mediator between the community, or rather the society, of Soviet mathematicians, and the individual Soviet mathematician. We come therefore to a rather intriguing turning point: is not an ancient Roman gold coin not a token of value, as well as an object of value, mobilising people in a way similar to that of an article? And if an article in the former USSR had a similar function to that of a gold coin, then who, or where, was its mint?

4.4 Banking Mathematical Intelligence

A mathematical proof, in order to be called “a proof” proper, besides the authorial party, needs always a counterparty, that is, its audience. Let us see a simple proposition along with its proof from a standard Soviet textbook [254, p. 24, my translation].²¹

Proposition. *Any infinite set contains a countable subset.*

Proof. Let M be an infinite set. Let us choose an arbitrary element a_1 in it. Since M is infinite, an element a_2 will be found in it, which is different from a_1 , then an element a_3 will be found, which is different from a_1 and from a_2 , and so on. Continuing this process (which cannot come abruptly to an end due to a “shortage” of elements, for M is infinite), we get a countable subset

$$A = \{a_1, a_2, \dots, a_n, \dots\}$$

of the set M . The proposition[, therefore,] has been proved. \square

We can now proceed to a more closer look to each step in the proof. It has to be always borne in mind that the above proof, as any proof in a published scientific article, is *intended to be read*.

- **Let M be an infinite set.**

The concept of a set is a primary concept. Theoretically it means, that it cannot be defined by other concepts; practically it means that the reader already knows how to produce himself or herself examples of sets, at least at this juncture of the textbook. The particular set is not referred to, only its property of being a set is mentioned. The third person imperative of existence, both in the above English translation, and the original Russian version, is a modern relic of mathematical

²¹Its partial translation in English is still a popular textbook for undergraduates in mathematics [253].

style originating from the corresponding ancient Greek one in Euclid's *Elements*.²²

Although formally it is not compulsory in a mathematical proof to name the generic set, it is customary, as well as necessary, to use symbols as names to help the readers “ease themselves” into the proof. The use of letters is a modern version, again, of the ancient Greek *lettered diagrams*, that is, geometric diagrams with important points on them named after a (Greek) letter (see [340, p. 12–67]). In the words of one interviewee,

[Symbols] are important, and a good notation, I find, helps to explain. [...] I like to rewrite, sensing my own way to understand it. [...] It helps; this is important for me.

The concept of infinity has been defined, in a rather informal way, earlier in the text [254, p. 21, my translation]:

Saying that a set is infinite, we mean that we can extract from it one element, two elements, and so on, and after each such step in that set, there are still elements left over.

We see, at this point, the participation of the reader in the definition of the infinite set, at least in an imaginary way. It is interesting to note that while the examples of finite sets provided just before this definition are, for instance, “all the molecules of water on Earth,” apart from mathematical ones, the examples of infinite sets are *only* mathematical ones: the set of natural numbers, that is, the numbers $1, 2, 3, \dots$, the set of all points in a straight line, or the set of all polynomials with rational coefficients. Mathematics, now, as a scientific discipline begins to emerge as a literary genre with its own fictional universe.

- Let us choose an arbitrary element a_1 in it.

This is, most probably, the most important part in a mathematical proof: it asks for the reader's own active participation in the process.

²²Ἐστω [Estō], “Let there be [one such and such],” third person singular; Ἐστώσαν [Estōsan] “Let there be [many such and such],” third person plural.

The reader has to “choose” an element, and then “name” this element a_1 . This is the point of picking up a pencil and start writing down the proof. Many mathematical proofs have a similar step: to prove is to repeat the whole, or parts of the, process yourself. How do we know, in this case, that indeed there is such an element to choose, and then, when we are certain that it exists, how can we choose it, and write it down in order to proceed to the next step of the proof?

At this point it starts to become apparent that mathematics, with its own literary universe, along with its mathematicians, is very similar to a (video) game in a virtual environment [1, p. 2, my emphasis]:

1. There is, first of all, the “*game-play* [of mathematics] (the players’ actions, strategies and motives);”
 2. Then there is its own “*game-structure* (the rules of the game, including the simulation rules);”
 3. And, probably, the most interesting of all is the mathematical “*game-world* (fictional content, topology/level design, textures etc.).”
- Since M is infinite, an element a_2 will be found in it , which is different from a_1 ,

We see again the participation of the reader in the process. Moreover, an element of a set, as well as a set, is another *primary concept*: the reader, somehow, or rather his or her “mathematical consciousness,” is assumed to be in a position to distinguish an element of a set; in practice, though, through class instruction, and then homework, the reader has acquired the ability to distinguish elements of sets, using a variety of methods.

- then an element a_3 will be found, which is different from a_1 and from a_2 , and so on.

The definition of an infinite set is implicitly called upon once again, and the reader is implicitly requested to participate and engage with his or her writing material.

- Continuing this process (which cannot come abruptly to an end due to a “shortage” of elements, for M is infinite),

The continuation of this process is a fictional one: there is this imaginary reader, who continues to pick elements from the infinite set in a fictional timeless universe. At this point it is rather evident the sneaky and underhand encroachment of imagination in a mathematical proof: there is no such natural reality or world in which such a process could ever continue *ad infinitum*.

- we get a countable subset

$$A = \{a_1, a_2, \dots, a_n, \dots\}$$

of the set M .

A *countable set* is a set that can be written in the above form, that is, as a sequence of numbered elements, whose subscripts run over all the set of natural numbers; an alternative written form for the above is $A = \{a_n\}_{n \in \mathbb{N}}$, where $n \in \mathbb{N}$ reads “ n is an element of the set \mathbb{N} of natural numbers.” By keeping the “surname” of each element to one common letter, that is, to “ a ,” the reader is facilitated into perceiving the infinite set A by the “ordered infiniteness” of the “names,” i.e. subscripts, of its elements, that is, $a_1, a_2, \dots, a_n, \dots$, and so on. The ellipsis, that is, the three dots “...”, after the generic element a_n add to the sense of infiniteness when reading this proof.

- The proposition[, therefore,] has been proved.

This is the official announcement of the ending of the proof. In English mathematical texts it is most often denoted either by QED [*quod erat demonstrandum*], which originates from the ancient Greek mathematical texts,²³ or by the “tombstone,” that is, “ \square ”, which is sometimes also referred to as “the halmos” (see [202, p.403]). The tombstone today is found also in Russian texts, due to the widespread use of the L^AT_EX document preparation system in mathematical texts.

²³ ὅπερ εἶδει δεῖξαι [oper edei deixe], “which had to be proved.”

The first interesting result about the previous “simple” proof is that in any infinite set there is always “embedded” a countable one. To a lay person this would seem indeed trivial, but to an imaginative mathematician that could lead to another more interesting question: are there any sets which are infinite but not countable, that is, that they cannot be written down as an infinite numbered sequence? As we will see in a later chapter, actually there are such sets, that is sets which are “bigger” than infinite sequences, as long as the concept of “bigness” has been defined in a mathematically appropriate way. This possibility of further exploration seems to be the “intrinsic” value of mathematical research, that is, the way economical value is being experienced by its economic agents. As one interviewee recalled:

I began to work as programmer in the Lebedev Institute. And after one year I realised that it is very boring for me, to program. At first it was very interesting. It was the first machines in the Soviet Union. [...] It was very interesting. I worked with physicists as assistant. I solved in computer some tasks for physics. [...] And after two years it was very boring for me. And I remember that I [...] decided to make a candidate’s dissertation. And I wrote three chapters of my dissertation.

Economic value has always to be experienced somehow by the economic agent himself or herself, but in the end of the day what is visible is not the experience but the action, or rather the material product of the particular social action, which can range from the sound of human voice, to a bodily movement, or even to the production of an artefact. In the case of mathematics research the product is the published book, or the published article. The previous proof on infinite and countable sets has another characteristic which has been many times cited as valuable: *mathematical rigour*. Mathematical rigour refers to the fact that we could do the proof based simply on the properties of being a set or being a member of a set. The infinite set itself, which could be an infinite set of numbers, functions, polynomials, or matrices, is totally irrelevant to the steps of the proof. Even the phrase “which cannot come abruptly to an end due to a ‘shortage’ of elements, for M is

infinite,” which has been inserted in parentheses in the proof, was redundant: it has been included only because this particular textbook was written for students, and not for mathematical researchers proper, in order to facilitate understanding.

The modern fusion of mathematical physics and theoretical physics is an interesting case in point in differentiating a mathematician’s experience of value from a physicist’s corresponding one. Yakov Sinai’s contributions in ergodic theory and statistical mechanics is one of these grey areas between mathematics and physics, in which Sinai and his students feel comfortable and at home both as physicists and mathematicians. But as to whether he is a physicist or a mathematician, Sinai himself has answered in a rather eloquent way [412, p. 407]:

Some years ago I proved a theorem in the theory of phase transitions. One Thursday, arriving at Landau’s seminar in the Institute, I chanced upon a talk about a paper by a Western physicist, in which the same result was obtained, but without precise expansions, estimates, and the like. A complicated question arises: Was it necessary to prove that theorem? Since then, in this situation and others like it, I have always answered this question affirmatively for myself, for one simple reason: the pleasure of a well-proved theorem is the highest possible.

Whatever, though, the individual experiencing of value may be, either in mathematics, or in some other field of social action, economic organisation thereof requires integration of this value into its wider community. The publication of the article serves exactly this purpose: it renders its mathematical value socially concrete by making this value *visible* to the community. Each economic agent of this community, that is, each mathematician, can himself or herself read it and borrow its results, or its methodology, or even both. Publication, therefore, of an article is a point in time where the author(s) of an article proceed to an *economic exchange* with the mathematical community through the economic fund of all the published articles. This economic fund is located, as one can easily infer, in the corresponding university

libraries. In mathematics, there are no property laws basically due to established custom, rather than due to any established local laws. Once Pythagoras proved his famous theorem, and once Einstein developed the general relativity theory, or, as we saw earlier, once Gelfand and Naimark proved their famous theorem in mathematics, the knowledge of these became automatically common property available to all. Copyright laws exist in other fields of human activity, but not in the field of mathematics.²⁴ Once, therefore, an article is published, then automatically it becomes the property of every active member in the community; it is an economic transaction simultaneously between the author(s) and the rest of the particular community of economic agents. The act of printing or photocopying, or even today scanning, an article from a scientific journal leads to another interesting conclusion: since every interested member of the community can materially acquire this article, if it is relevant to their research, we can say that a published article acquires the potential, after the exchange, to circulate. Economic circulation of scientific value has been measured in Western science through the *impact factor indicators*: bibliographic citations are officially declared evidence that the article author(s) “got their hands on materiality” of the the cited articles (for more on the impact factor see [165]). We saw, though, earlier, in the text, that the impact factor does not necessarily imply economic value in mathematics, although it does have its uses.²⁵

The distinct importance, in other words, of a mathematical article is its being a *material*, and therefore *visible*, token of value. Publication is the exchange transaction which makes the value of mathematical knowledge visible to the community. Circulation, now, can be seen as the act of an individual scientist claiming his or her customary right to the visibility of a published article’s value. In other words “[d]isplay and exhibition as realized in exchange [...] verify the resources of giver and receiver: objects and object qualities become definitely associated with particular persons” [153, p. 24]. But this visibility property of exchange is very reminiscent of money [394, p. 223, my emphasis]:

²⁴And moreover copyright is a modern Western legal-contractual invention.

²⁵Although the Pythagorean theorem has been, and is being widely circulated, its publication value today is rather negative: instead of going unnoticed, it will receive a lot of negative criticism.

Money serves to distinguish its *possessor*, yet to do this it must be “seen” somehow by others. Some sort of public display of a money sign is needed to *reveal the potency* of money, or to put it in another way, money, as a quantitative and relative value, needs to transform itself into a quality, or at least into an object signifying quality to others.

Money, in other words, is useless, that is, it cannot mobilise agents into a financial transaction, unless it is being somehow produced on the spot, brought forth into the light. Even in a transaction as simple as in a superstore when using a debit, or credit, card, the visibility of the money owner’s financial value is displayed electronically: the PIN number of the card ensures and verifies the financial identity of its possessor, and then the electronic banking communication network verifies the financial potency of the possessor. There is no way that a financial agent will attempt to proceed to a financial transaction with another financial agent, unless the latter has somehow exhibited, either explicitly or latently, his or her *financial potency*, or at least some kind of potential financial potency. The important thing about visibility is that “[s]eeing, in other words, constitutes not only knowledge but also control: seeing performatively asserts propriety rights” [153, p. 25]. This is, for example, what happens when some tax office discovers, that is, “sees,” a case of tax evasion, and then asserts and imposes extraordinary property rights, besides ordinary ones, on the evaded sums of money through fines.

Contrary to money, though, a mathematical article can be exchanged only once: when it is published it becomes common property, which any member of the scientific community can claim and acquire. On the other hand, since an article is indeed valuable due to its contents, at least at the time of its publication, we can say that it is a commodity to be “consumed,” that is, to be read and understood. This valuable commodity, which both in the common, and the academic parlance, is called scientific knowledge: here it will be called *intelligence*. In keeping with the concept of *generalised media of exchange*, (mathematical) intelligence will be, rather loosely, defined as “a generalized *capacity* controlled by any acting unit [i.e. any economic agent] to contribute to the implementation of cognitive values through knowledge,

through the acquisition and use of competence, and through the pattern of rationality” [351, p.70-71, emphasis in the original]. An article, in other words, is commodity money, both as having value in its content, and as being a token of value. It is not, though, a fungible commodity, since there is no officially acceptable accounting or measuring system to create divisions or multiples of it; at least not in the case of the Soviet Union. Actually there is one aspect of a scientific article which has not yet been mentioned in the text: the scientific journal under the auspices of which the mathematical article is published. Earlier in the text we saw that the Soviet mathematical community was organised as a gift economy, the gift being the scientific article. Then we saw that the mathematical article is an heirloom of its authors, which becomes shared property after its publication. Then after the publication all the members of the scientific community could lay a claim of acquiring the article. What was missing from the equation was the *authority of the journal*, behind which were its editors. One peculiarity of Soviet mathematics, and Soviet science in general, was its lack of a peer-reviewing process. The journal editors were officially the “peers” who were usually academicians, that is members of Soviet Academy of Sciences [5]. An article, in other words, was the two sides of a coin: the “head” was the scientific journal, that is, the “state” (or rather banking) issuing authority guaranteeing its scientific validity, and the “tails,” was the article itself, that is, the underlying valuable commodity being used by the “market,” (or rather the community) of scientists (see [206]). An article, thus, was not only a gift, but also a coin, its mint being some scientific journal. Or to be more exact: the existence of authors between the article content, being the commodity, and the authority of the journal, signifying its token value, made the article something between a gift and a coin, a gift-coin of mathematical intelligence.

On the 22nd of December 2006, Science magazine for the first time in its history had on its cover a mathematical theorem as the “Breakthrough of the Year”: the 100 year old unsolved *Poincaré conjecture* had finally been solved [304]. The Poincaré conjecture was one of the most difficult problems resistant to any attempted solution in mathematics. It was first formulated in the beginning of the twentieth century by the French mathematician Henri

Poincaré, its solution was conjectured by him, and since then it had remained unsolved. In 2002 and 2003, the Russian mathematician Grigori Perelman, at the time employed in the St. Petersburg branch of Steklov Mathematical Institute, published three online preprints, 39, 22, and 7 pages, claiming that he had sketched a proof of the conjecture in his three papers. A sudden public stir was created in the mathematical community. Perelman was called by various universities in the USA to present his theoretical programme. In one of these papers he wrote [356, p. 1]:

I was partially supported by personal savings accumulated during my visits to the Courant Institute in the Fall of 1992, to the SUNY at Stony Brook in the Spring of 1993, and to the UC at Berkeley as a Miller Fellow in 1993-95. I'd like to thank everyone who worked to make those opportunities available to me.

Three independent groups were set up to verify the proof. Bruce Kleiner and John Lott, working at the University of Michigan, submitted on 25 May 2006 the first version of an online preprint of 192 pages filling in the details Perelman had left out. In June 2006 Zhu Xiping of Sun Yat-sen University in China and Huai-Dong Cao of Lehigh University in Pennsylvania published a paper of 328 pages on the Asian Journal of Mathematics. In the abstract of the paper they said that their “proof should be considered as the crowning achievement of the Hamilton-Perelman theory of Ricci” [222, p.165]. On 25 July 2006 John Morgan of Columbia University and Gang Tian of the Massachusetts Institute of Technology submitted the first version of an online preprint of 473 pages providing a detailed proof of Perelman’s papers. In the meantime, the Poincaré conjecture had been proclaimed as one of the Millennium Prize Problems, that is, one the seven problems that the scientific board of the Clay Mathematical Institute had established with the purpose of recording “some of the most difficult issues with which mathematicians were struggling at the turn of the second millennium” and of recognizing “achievement in mathematics of historical dimension” [82, p. vii]. Moreover, “[a] set of rules was established, and a prize fund of US\$7 million was set up, this sum to be allocated in equal parts to the seven problems.”

In August 2006, during the International Congress of Mathematicians held in Madrid, Spain, Perelman, along with other three mathematicians,²⁶ was awarded the Fields Medal, the most prestigious prize in the field of modern mathematics. The Fields Medal, named after the Canadian Mathematician John Charles Fields who established it in 1936, is awarded every four years to two to four mathematicians who are under forty years old on the day of the award ceremony and “is intended not only to recognize results already obtained, but also to stimulate further research” [327, p. 3] (see also [389]). According to John Coates of Cambridge University “[t]his saves us a lot of trouble [since w]ith the Nobels many of the old men on the committee are possible winners themselves, but that’s not the case with the Fields” [51]. Perelman declined to accept it, for the first time in the history of the award. On 28 August 2006, a few days after the Fields medal award ceremony, a newspaper article was published, attempting to narrate the full story behind the proof of the Poincaré conjecture [337]. One of the authors of the article was a well known journalist who had written John Nash’s biography, a famous American mathematician struggling for most of his life with schizophrenia, and it had later turned into a Hollywood movie [336]. The authors of the article had conducted an interview with Perelman in St. Petersburg, as well as many more with mathematicians all over the world involved in the Poincaré conjecture. Sir John Ball of Oxford University, chairman-president, at the time, of the International Mathematical Union, had gone to St. Petersburg himself earlier in the year to speak to Perelman as to whether he would accept the Fields medal or not. Perelman explained to the journalists that the Fields medal “was completely irrelevant for me; [e]verybody understood that if the proof is correct then no other recognition is needed.” After the publication of the newspaper article a rather heated controversy followed in the mathematical community: the article depicted modern mathematics as a community ridden with power struggles and infested with machinations over prizes and awards. Lawsuits were threatened to follow, as well as denials of statements were issued allegedly made during the interviews before the publication of the article (for a fuller account see [168]). Perelman’s case started to attract wider media attention, and finally, in 2010 the unexpected

²⁶One of them another Russian, Andrei Okunkov.

happened: on March 18, 2010 the Clay Mathematical Institute announces its decision to award 1 million dollar to Perelman for solving the Poincaré conjecture; three months later, Perelman, who had already quit his post in the Steklov Institute in St. Petersburg in 2005 and had remained unemployed since, declines the prize [391].

Perelman at an early school age had been to the Leningrad Palace of Pioneers, in the mathematics club for schoolchildren [168, p 18]. The Young Pioneers were the communist equivalent of the Western Scout Movement. These were organised around some common-interest topic, such as science, sport, art, politics, etc. They organised extracurricular activities that the communist party considered important. In the case of mathematics, for example, as well as in other fields, “there were camps for Young Pioneers in the countryside where students could go on their summer vacations” [308, p. 403]. Later on, Perelman joined the Physical-Mathematical Lyceum, No. 239, in Leningrad (today St. Petersburg). There were many such specialised high schools in the former Soviet Union. Quite often “11–12 hours per week were allocated for mathematics, which was twice the usual amount [of non-specialised schools]; furthermore, the course content was far more intensive and challenging” [234, p. 267]. One of the interviewees, employed full-time by an American university at the time of the interview, commented on the time he was a pupil in Lyceum 239: “They were school teachers, but very good; so in America they would be full professors at universities. The level of competence so high.” Perelman, after high school, entered the Leningrad State University, and then, after graduation he became a PhD student in Steklov Mathematical Institute, Leningrad Branch.²⁷ During all these years he had coaches, advisors and collaborators who were close to him, and they all had a strong influence on him [168]. He submitted his candidate’s dissertation in 1990, just a year after the unexpected by everybody collapse of communism.

In 1996 European Mathematical Society (EMS) awarded Perelman its prize but he refused to accept it. The EMS prizes, in a similar fashion to Fields medals, are awarded every four years to ten mathematicians not older than 35 years at the day of the award ceremony. One could easily say that this

²⁷Today St. Petersburg Branch.

was probably the first sign for what was to follow. While the solution of the Poincaré conjecture stirred the international mathematical community, the 10 million dollars of the Millenium Problems prize brought widespread media attention to Perelman. There were even reports of Russian paparazzi breaking into his mother's flat to get interviews from her, and Perelman bursting into profanities. Gessen has a whole chapter on her book attempting to explain his behaviour as being most probably a behavioral manifestation of Asperger syndrome. People with Asperger syndrome,

are typically motivated to interact with others, but find themselves socially isolated because of their odd communication style, which is often overly formal and may take the form of an in-depth monologue about a topic of special interest regardless of whether their interlocutor is interested or not [464, p. 3].

Leaving aside the specious scientific validity of an “attempted diagnosis” of a clinical syndrome by a lay person, there is another aspect of the incidents surrounding Perelman's case: this was more of an instance of Soviet mathematics, rather than an isolated case of an alleged madman. Perelman considered money, in its economic financial value, as just a means to produce the economic value of an article. He had saved money from his previous appointments in American universities to fund his own research on proving one of the Holy Grails of mathematics: the Poincaré conjecture. The gift he would contribute to the mathematical community had been only projected and imagined for a century: it was a gift that only a god could donate. The Fields medal, therefore, was trivial when compared to the Poincaré conjecture. His reclusiveness, which was widely mentioned in the media, actually kept him close only to the gift economy of Soviet mathematics: his tendency for isolation prevented him from integrating with the new Russia, the Russia of American capitalism. He *had* bonded with his parents and his sister; he *had* bonded with his mathematics coaches during his school years; he *had* bonded with his advisors and supervisors during his graduate and post-graduate years. He had adopted, in other words, the gift culture of Soviet mathematics. With the transition from the Soviet Union to a capitalist Rus-

sia, he simply did not adapt to the new economic culture which fused the scientific economic value of published articles with the financial economic value of banknotes and bank deposits. His reclusiveness, in other words, prevented him from integrating with the new post-Soviet economic system; he was not devoid of an economic cultural system. Grigori Perelman, most probably, was the last great Soviet mathematician proper.

Although Perelman was awarded the Fields medal in 2006, the announcement for the Millennium Prize on the Poincaré conjecture came out on 2010. In the rules for the award of the prizes, the Clay Institute specifically mentioned that

a proposed solution must be published in a refereed mathematics publication of worldwide repute [...], and it must also have general acceptance in the mathematics community two years after [82, p. 153].

Perelman published his proposed solution on arXiv.²⁸ arXiv is an electronic repository of preprints, that is, of drafts of scientific articles, not yet published in a peer-reviewed scientific journal (see [227]). It was created by the physicist Paul Ginsparg in 1991 in order to facilitate the exchange of physics preprints among American universities, in the beginning, and then expanded to other disciplines, such as mathematics, astronomy, computer science. The repository is hosted today in the servers of the Cornell University. Although its content is not properly peer-reviewed, there are some moderators, in lieu of editors, who check whether the preprints fall into the submitted area, and they may recategorise them. In the Soviet Union, the editors in a scientific journal were actually more of moderators, as in the arXiv repository, rather than peer reviewers proper. In an interview conducted in 2006, when asked why he had not published his proof in a peer-reviewed journal, Perelman said that

[i]f anybody is interested in my way of solving the problem, it's

²⁸Where the capital X is pronounced as the *ch* in the Scottish word *loch*, which is the pronunciation of the Greek letter χ .

all there – let them go and read about it. I have published all my calculations. This is what I can offer the public [295].

Finally in 2008 Bruce Kleiner and John Lott published the full and complete proof of the Poincaré conjecture in *Geometry and Topology*, a well known peer-reviewed journal in mathematics [247].

The delay of the Clay Mathematical Institute award poses another problem in mathematics: that of an increasing complexity in scientific research. More people had to be involved in the verification of the proof; the scientific validity of the non peer-reviewed arXiv repository versus the peer-reviewed journals could be easily disputed. Soviet mathematics had, in fact, gone a long way ahead since the first turbulent period Stalinist Russia: it had become a well established and respected partner along with other countries in the massive production of mathematical theorems. As one interviewee remembered from his postgraduate years:

It was decided to organise a very good probability school in Bakuriani. Bakuriani this is a small village [...] and you can not only visit the conference, but [also] you can spend some time by mountain skiing.²⁹ [...] For example when I was a PhD student, I visited this school two times as a PhD student, and it was the time when I first learned such a person as Sinai, Shiriyayev, Prokhorov, Sazonov. So it was very helpful for me because you can have conversation with all these people. [...] So it was extremely important for young generation. One day in this Bakuriani school Prokhorov said something very interesting. He tried to explain how probability theory flourished in the Soviet Union, and he said that when he was 40 years [old], he knew every PhD [student] in the probability theory in the Soviet Union. Now [that] he is 60 he didn't even know every doctor in probability

²⁹These were the *All-Union Winter Mathematics Schools-Colloquia in Probability Theory and Mathematical Statistics* held every year in Bakuriani, a skiing resort in Georgia. The first took place in 1967, and the last in 1990, one year before the dissolution of the USSR.

theory.

The development and growth of mathematics had gone from the personal level to the impersonal. With the massive acceptance of postgraduate students the complexity of mathematical research had massively increased, as well. The human individual, whether modern, premodern, or prehistoric, still remains tribal in its perception of the world: she perceives the world in general through stories, personal relationships, and artefacts. The main artefacts allowing imaginary access to the whole of the scientific world in the case of Soviet mathematics, and of science, in general, was, and still remains, the published scientific article. Still the production rate of published articles was increasing to an unprecedented level. Even if we supposed the existence of a “super-mathematician” understanding every field and subfield of mathematics, it would still be physically impossible to read every one published article: one lifetime most probably would not be enough. The Soviet scientific social system of research in mathematics, in general, and of probability theory and mathematical statistics, in particular, was creating by itself “a problem of overload and of constantly threatened instability” [297, p. 6].

During a conference held in Moscow, the author had at some point a brief discussion with one of Vladimir Arnold’s students, who is a leading researcher in algebraic geometry. During this discussion the mathematician mentioned the fact that he could not understand some of the papers presented in the conference, and moreover, it was becoming difficult for him to keep up with the latest developments. The Soviet system of research was very demanding in many respects. The university graduate, if she wanted to continue to a Candidate of Sciences degree (*Kandidat Nauk*), the actual equivalent of a PhD in the English-speaking universities, was expected to have at least two articles published at the time of the application. Not to mention that the university was five years, not four, as in the Western European universities, and certainly not with a majoring subject as in the American universities: the student had already chosen her major from the first year. Moreover, a candidate’s dissertation was not, in general, enough: the doctor’s dissertation was an extra requirement for those with more research aspirations. Despite its ostensible similarity with the German and French habilitation, Soviet, as well as

modern Russian, doctor's dissertation was submitted in most cases approximately after ten years of published research and the Doctor of Sciences degree (*Doctor Nauk*) was awarded after demonstrating a major breakthrough in the field. Moreover, the awarding body both of a candidate's degree, and of a doctor's degree, was the Higher Attestation Commission for the whole of the USSR. German and French Doctor's degrees, on the contrary, are awarded by the university in which the dissertation, or habilitation, has been submitted. As a result, there was more *trust* in the research judgement, competence, and the ability of the individual mathematician, which compensated for the lack of proper peer-review. In the Western universities, on the contrary, there is more trust in the impersonal scientific journal, and its impersonal processes of validation, rather than on the author herself.

The problem of trust for a social scientist is how can one measure trust. In fact, systemic trust is an imaginary economic resource and so far there is no method to measure it. What is measurable, on the contrary, is the *lack of trust*: when there is a widespread rumour, for example, that a commercial bank is facing financial problems, and is going soon to file for bankruptcy, then in the majority of the cases a *run on the bank* follows which is the most visible evidence of lack of systemic trust.³⁰ Soviet mathematics, in particular, and Soviet science in general, enjoyed worldwide systemic trust for its scientific research operations, especially after being the first country to send a man in space. In the case of mathematics, this trust in the individual researcher, and by extension in the whole Soviet social system of mathematical research, compensated for the increase in its complexity: it was not necessary for the individual researcher to check and verify the proof of every published theorem. While, in other words, the inherent tendency of scientific research to increase the complexity of the scientific social system, the trust accorded to the individual researcher by her colleagues, backed by the great demands placed on each individual researcher by the scientific system itself, reduced, in its turn, the complexity of the whole Soviet scientific social system.

In the Roman legal system there were two special kinds of contracts which

³⁰Galileo, on the other hand, attempted quite successfully to manipulate different mechanisms of systemic trust in medieval Italy to his own benefit (see [25]).

have remained rather unchanged even today:

the contract of deposit (*depositum* in Latin) is a contract made in good faith by which one person – the depositor – entrusts to another – the depositary – a movable good for that person to guard, protect, and return at any moment the depositor should ask for it [420, p 4].

The usual example of a deposit contract is when the depositor entrusts a diamond to a depositary, such as a bank, to guard it until the depositor asks the diamond back. Depending on the depositary, the execution of the deposit contract might entail a certain charge for the service of guardianship, that is, a commission. If the good is a fungible one, that is, a good whose quantity and quality can be replaced by another good of the same quality and quantity (the *tantundem*), then we have the *irregular* deposit contract. In the irregular contract the depositor may entrust a fungible good, such as oil, and then if she asks it back after some time, the depositary returns the same amount and quality of oil, that is, the *tantundem*. Returning the same amount means that the depositary has some commonly, or at least mutually, acceptable counting, or rather accounting, method of a certain natural property of the oil, such as weight, or volume. Returning the same quality means that there is some mutually acceptable way of identifying quality such as chemical ingredients contained, or point of origin. Gold coins stamped by the same issuing authority are always fungible: a depositary can always return upon request any gold coin that has been issued by the same issuing authority as that, which had been given in the first place. Moreover, if two gold coins, are genuine, and have the same weight, they can be considered as equivalent in personal transactions anyway, since gold, and therefore gold coins, has always been considered a commodity of universal value, independently of the issuing authority.³¹ It has to be clarified, though, that the deposit contract, whether regular, or irregular, does not transfer the ownership of the deposited good: the depositary cannot sell and then buy back a

³¹Nature could be said to be the issuing authority in this case, an authority beyond all human authorities.

diamond, or an amount of gold coins that have been entrusted to her. The idea of a deposit contract is to attract depositors who are *risk-averse*, that is, being a depositor means that one wants to avoid risk. These were the initial reserve banking systems in the Roman times. Today we have *fractional* reserve banking: in the case of money, the banks acquire temporary ownership of the deposited amount, and can then invest it, apart from a fraction, usually 10%, of the total amount deposited in them which are obliged to retain for everyday transactions with the public. In case of a bank run, the central bank provides the rest 90% of the money expected to be missing from the private banks. The modern banking system, in other words, is the result of continuous violations of the original Roman contract of depositum (see [420, p. 1–166]; for the Roman depositum contract see also, the rather demanding, [476, p. 188–229]).

A banking institution, in general, consists of three “sections”: the directors, the accountants, and a vault. If we consider the scientific journals as the issuing authorities of published articles, that is, the banking institutions minting the coins of science, then the editors are actually its corporate directors with *fiduciary duties*, a custom dating back to the Roman period:

Roman law provided for certain situations where one party acted for another not as an agent but in his own right. This was the concept of *trusteeship*: the trustee held a right in somebody else’s interest; on account of the *fiduciary relationship* he was bound, however, to safeguard these interests of the *beneficiary* [476, p. 50, my emphasis] (see also [119]).

The journal editors, in fact, are *entrusted* with certain duties. The Soviet mathematicians, who were editors in scientific journals, in other words, had the fiduciary duties of a “capitalist” corporate director. The important thing, though, to distinguish at this point is the sociological aspect of *contractual customs*. One of the oldest contractual customs, which actually dates back to the Babylonian and Assyrian empires is debt (see [181]). Contractual customs, in other words, are part and parcel of civilisation, or at least of societies with writing systems. Since modern European culture, as well as imperial

Russian and Soviet ones, is a legal descendant of the Roman legal culture, contractual customs have been backed by artefacts, i.e. written contracts. The fundamental form of contract in the Roman republic was the *stipulatio*, also known as *contractus verbis* that is, the verbal contract [476, p. 68–94]. In the late Roman empire, a new development was the *contractus litteris*, that is, the literal (written) contract. Justinian put an end to the *contractus litteris*, but it was later “rediscovered” by medieval lawyers [476, p. 546–547]. There is, therefore, the Roman social custom of appointing the directors of a scientific journal, and then this is backed by the additional (Roman) social custom of recording the social event of the appointment on a piece of paper.³² The names of the editors of each Soviet mathematical journal was always mentioned on the first pages of the journal, as well as in the internal documents of the journal. No matter what the Soviets could claim as to the revolutionary essence of their society, in terms of some of their contractual customs they were more Roman, than they would have, most probably, believed themselves to be. The vault of the scientific journal was the room with the journal archives. But, most probably, the most important “section” in the journal were its accountants: the individuals who decided whether an article should enter the “vault of mathematics” or not. In the Western universities these accountants were, and still are, the peer-reviewers: they decided as to the scientific validity of an article.³³ In the case of the Soviet Union, the editors, were not only the directors, but also the accountants of scientific validity, as well as any senior member of the scientific community, who could identify gaps in the proofs. We see now, that actually the authors of a scientific article were actually the depositors of the banking system of Soviet science. Moreover, since by custom there were no, and still are no, property, or copyright, laws as to the contents of an article, a deposit of one scientific coin, that is, a published article, rendered automatically all the members of the community depositors of the same coin. In addition the deposit made was a regular deposit, rather than an irregular one, since scientific articles

³²Which could also be an Egyptian scroll or a pdf file in a computer file; that depends on the technology of artefacts, and what types of artefacts are considered as valid in a particular civilisation.

³³That is, in simple terms, whether a mathematical theorem is true or not.

are not fungible, at least in science, and at least up to today's publishing accounting customs.

The university library has also played a very important role in mathematical research. It is actually the external memory of the scientific community. The university library, has, in fact, been the material vault of scientific intelligence: it contains the actual paper articles, and books of mathematical knowledge. Destruction of a library is a major event in any academic institution. The most infamous destruction of a library, probably, has been the burning down of the legendary Alexandrian Great Library of the Hellenistic period, an event that reverberated all over the academic world for many centuries to come (for a general history of libraries over the ages see [335]). Even the Third Reich has been stigmatised, among other things, in the public opinion for institutionalising the public burning of books: the feeling that a book belongs to everybody had been inculcated for many centuries in the West, as well as later in the USSR. A researcher can borrow books and articles to “use”, that is, to read and understand them. Then she can extend many times the period of borrowing; still, though, at some point she has to return the books, and the articles if she has borrowed whole journals. If we want, in other words, to understand the social customs in regulating the university library as external memory, we have to go back again to Roman contractual law and then return back to modern banking practices. The Roman *commodatum* contract

refers to a real contract made in good faith, by which one person – the lender – entrusts to another – the borrower or commodatary – a specific item to be used for free for a certain period of time, at the end of which the item must be restored to its owner.[420, p. 2].

The *mutuum* contract, on the other hand,

refers to the contract by which one person – the lender – entrusts to another – the borrower or mutuary – a certain quantity of fungible goods, and the borrower is obliged, at the end of a specified

term, to return an equal quantity of goods of the same type and quality (*tantundem* in Latin) [420, p. 2].

A typical example of a commodatum contract, or rather of commodatum contractual behaviour, is when someone lends her car to a friend, the commodatary, for a specified period of time, such as a week, and then, after having used it for his business, the commodatary returns the car intact after a week. When someone lends a basket of cherries, on the contrary, to a friend for a week, with the provision that her friend, the mutuary, returns a basket with the same quantity and quality of cherries, after a week, then it is a mutuum contract. Contract here refers to the social custom of mutual agreement as to the particular terms of that contract. The additional Roman custom of writing down the contract on some kind of artefact, such as a piece of paper, or a scroll, will not concern us. Since scientific articles are not fungible, the mutuum contract is not related to scientific research. The commodatum contract, as well as the mutuum one, is actually a loan contract: the commodatary, that is the lender, acquires full ownership for the period of the loan. In the case of a university library, ownership refers to the material vehicle of a book, that is, the white printed pages, the binding, the hard covers, and so on. Ownership of the book means to be in a position to materially carry it with the hands, to open its pages with the fingers, browse for as long as one likes, and so on. The mathematical content, though, of a book can be used, and cited as having being used, in any other later article, or book.

A special case of a commodatary was actually the postgraduate student (*Aspirant*). The officially prescribed and accepted period of postgraduate study was three years. One interviewee, in particular, finished his dissertation earlier:

I was not paid, but I was his aspirant. I was working in a research institution in Novosibirsk itself, but I made [the] dissertation in one year. Yes, yes I did. For some purely technical reasons, I was unable to defend it for two more years. [But t]he results were in one year.

This was one of the infinite examples of the frequent irrationality of Soviet bureaucracy: in the end of the day it was not the Soviet system's fault that this particular interviewee as a student had proved to be more prepared and more talented for postgraduate research than the system itself had anticipated; he should have been more frugal and more reserved in his aspirations. Most of the interviewees mentioned their postgraduate years as a period of freedom. What is actually more important, especially in the case of postgraduates, is how the banking system of mathematical intelligence was actually organised around credit expansion. The purpose of a commodatum contract, and of any bank loan in general, is about how much is the charge of the loan, that is, how much is its interest. A postgraduate student before entering officially the postgraduate research, had already been a depositor in some bank by having published already at least two articles as an undergraduate. Then, at the proposal of her supervisor, by becoming a postgraduate researcher, as an *aspirant* she could "borrow" resources from a university library, and at the end of the three years he would return the economic resources he initially borrowed, as well as an amount of interest, that is, added economic value: the publication of her dissertation. And any outsider could clearly see that this credit process was conducted with Renaissance Italian double-entry bookkeeping in mind (see [276, 349]): the university library, or rather the university institution itself, was crediting its vault with resources to the postgraduate student for three years, and at the same time debiting its vault with the obligation of the student to return the borrowed resources as well as the added value of a published dissertation. On the opposite side, the student was debiting herself with the economic scientific resources of the university and crediting herself to the university vault with her double obligation both to return the resources she had borrow, as well as to deposit to the vault the added economic value of a published dissertation. The public defence of the dissertation in front of an audience of academics was the accounting procedure, in which a collective decision was made as to whether the submitted dissertation would be acceptable as added value to be debited to the scientific vault or not. Such a system of credit, and actually any banking system with credit expansion, was based on cross-institutional, or rather systemic, trust, that is, its depositors never questioned, or doubted about it; in fact,

all the interviewees, especially those in the USA, who had come into contact with another university social system, when asked during the interview, they still considered the Soviet system as superior to others. One interviewee, for example, who was a professor in a North American university, was asked if he considered accidental the fact that during the last sixteen years, in each one of the four International Congresses of Mathematicians that were organised in 1998, 2002, 2006, and 2010, there was at least one Russian mathematician who was awarded a Fields medal; his answer was as follows:

No, no, no. There was a wonderful culture, and still, it spread [...] not at all. Well, accidental, it was a historical moment, when the Russian school was really prominent, and it had big advantages over other schools, even the famous French school, and we have our Bourbakis³⁴ there [...] but it's difficult to separate it from the history, from the society.

Moreover, many Russian mathematical scientific journals were translated into English intact, in spite of their lack of peer-reviewing in the Western sense: in other words, the Soviet banks of mathematical intelligence were indeed enjoying the trust of their Western counterparts.

4.5 The Estates of the Soviet Realm

During the middle ages, a certain model of society was widespread among the intellectual, and political elites of western Europe. According to this model the medieval western European society comprised three “social orders”:

there was a distribution of tasks in the city of God (identified here with Christianity): the *oratores* (those who prayed), the *bellatores* (those who fought) and the *laboratores* (those who worked). They all lived together and could not imagine being separated. Each group supported the other two [39, p. 59–60].

³⁴Bourbaki was a collective author name of a group of French prominent mathematicians who were publishing many influential books on very advanced mathematics.

These three orders were the three *Estates of the Realm*. The First Estate, in order of importance, was the clergy: they were the ones who espoused and proliferated this social model as the model ordained by God Himself. The Second Estate was the nobility: their social function, as perceived in this model, was to protect God's realm from any potential intruders, who would threaten its existence. The Third Estate were the commoners, those basically excluded from the previous two, that is, artisans and peasants: their (perceived) mission had to do with what we would be calling today the "logistical operations of the God's realm." The interesting thing about the three estates, in terms of social mobility, was that, although the second and the third ones were hereditary, the first, and most important one, was, in principle, open to all: anyone joining the ranks of the Church could reach the highest possible social position during a single lifetime. This social model of medieval Roman Catholic historiography, which was "[i]nspired more or less remotely by St. Augustine's *City of God*" [39, p. 59], passed on later to the monarchy of the French *ancien régime*, since

the fitting of social relations into ternary structures made it possible to integrate those into global structures, which extended over the entire visible and invisible universe [132, p. 2].

The social order of the Estates of the Realm led to the creation of the French legislative body of *Estates General*, which met until 1614: it was "the last to meet before 1789, and the ability of the French kings to govern without a national representative body was a defining feature of absolute monarchy" [427, p. 140]. The French *ancien régime*, with the appearance of new socially visible types of commoners, such as the merchant class, became less rigid in terms of social mobility. The second estate of the nobility, though, still gave off social prestige:

Throughout the eighteenth century concern for reputation and status continued to exert a strong influence on the use of wealth ... [S]tatus was able to co-opt wealth by establishing standards for the acceptable use of money [59, p. 45].

The Estates General was reconstituted, and representatives from all three estates were called upon to participate in 1789, in an attempt to control to civil unrest that was about to break out in France, under the name of National Assembly. During the French Revolution the Estates General was finally dissolved as a legislative body of the three estates, and reconstituted as National Assembly.

In medieval England this division into the three social estates lead to a bi-cameral parliament. It originated from medieval king's councils, which were becoming more established after the granting of Magna Carta. In the beginning only the nobility and the clergy were participating in the official meetings with the king. But later on,

the “commons” (the knights and burgesses) came to be summoned with much greater frequency, and after 1325 no assembly which excluded representatives of the Commons was described as a parliament [173, p. 16].

This compulsory “inclusion of representatives of lower status” lead the Lords, that is, the nobility and the clergy, “to insist upon their exclusiveness as a separate ‘estate’ in parliament” [173, p. 16]. In medieval Scotland “ ‘the community of the realm’ – a phrase that would sum up the whole political nation – was often understood to mean parliament” [176, p 301]. In 1357 the first official recording of the phrase “three estates” was made and this

threefold division into estates reconciled two existing sets of twofold divisions: clergy and laity on the one hand, and ‘lords’ and burgesses on the other” [176, p 302].

In 1707, with the Treaty of the Union, the British parliaments were unified into one, as the Article III stated:

That the united Kingdom of Great-Britain be represented by one and the same Parliament, to be stiled the Parliament of Great Britain [205, p 170].

The Treaty of the Union, in contradistinction to the developments with the French Estates General, was the result of many political interactions and negotiations³⁵ among factions and interest groups belonging to a self-described society of three estates. The modern British House of Lords, an institutional relic of medieval Britain, still comprises the modern versions of the first two medieval estates: the Lords Spiritual, that is, the archbishops and bishops of the Church of England; and the Lords Temporal, that is, representatives of the modern British nobility (see [35, p. 14]).

In the Russian middle ages the situation was much more complicated. After the Great Schism between the Roman Catholic Church and the Eastern Orthodox Church, the Roman Catholic Church had no influence on early medieval Russia. Medieval Russia had converted to the religious doctrines of the Eastern Orthodox Church, and the three-estates social model of the Roman Catholic Church was not initially adopted: by 1054 the Patriarchy of Constantinople and the Holy See had excommunicated each other. During the time of Peter I the Great, Russia “consisted of numerous, small groups and lacked collective terms for legal aggregation” [158, p. 14]. Then Peter the Great introduced the Table of Ranks

“which stressed the link between service to the crown and noble privilege, and created the rule that officers and civil servants acquired noble status automatically upon reaching defined ranks” [291, p. 229].

The Table of Ranks introduced “the *chin* [i.e. rank]³⁶ system, or system of rank ordering and niche assignment” [41, p. 2]. Although the *chin* system of the Table of Ranks expanded and refined the social base of the nobility, it left out large portions of the general population. In the seventeenth century an influential Russian thinker

described an ideal society strictly divided into classes or estates

³⁵Or political conspiracies and machinations, depending on the narrative a historiographer might prefer.

³⁶Or *chinovnik* [i.e. official, functionary].

(*soslovie*)³⁷ — nobles serving the state, merchants dedicated to commerce, artisans pursuing crafts and peasants ploughing the land [203, p. 121].

In the last two decades of the eighteenth century two new legal terms were introduced: *sostoianie* and *zvanie*. While *zvanie* “tended to be closely associated with specific occupational groups” [158, p. 16, n. 17], *sostoianie* “remained the basic legal term until the end of the *ancien régime*” [p. 16]. Historiographers of Imperial Russia at the time found social-semantic similarities of *soslovie*, the Russian word most commonly associated today with the social estates, with the corresponding European terminology:

Soslovie (*ordo* or *status*, Fr. *état*, Ger. *Stand*) is a term of state law and marks a well-known rank of political institutions. We call *soslovie* the classes into which society divides in terms of rights and duties [248, p. 225, my translation].

On the other hand, the Russian *soslovie* “did not play a political role equivalent to that of the French *États* or German *Stände*, but they did share important features with these groups” [462, p. 245]. A dictionary published in 1847 by the Russian Academy of Sciences described *soslovie* as “a category of people with a specific occupation, distinguished from others by their special rights and obligations” [158, p. 19]. At some point the legal definitions and the cultural origins of a *soslovie* became too complicated to organise imperial Russia into manageable groups by the state: Cossacks, a historical east Slavic ethnic group, had the legal-cultural designation of a separate *soslovie* while at the same time they could belong to the military *soslovie*; physicians belonged to the medical *soslovie* and lawyers were members of the juridical *soslovie*. On the other hand a large number of

underclass of beggars, wanderers, the homeless and the diseased ...evaded the *soslovie* (estate)-based categories of nineteenth-century statisticians [... who additionally] ignored the emergence

³⁷Plural of *soslovie*, that is, estate.

of significant new social groups, most notably middle classes and workers [143, p. 272–273].

Until the end of the Russian *ancien régime*, in spite of the state's effort to expand or particularise the *soslovie* paradigm, “social identities remained in a high degree of ambiguity and flux, oscillating between legal estate, economic status, and occupation” [158, p. 34]. When the Bolsheviks came to power in 1917, they issued a decree abolishing all social estates and state ranks.

In the British, the French, and the Russian cases we can clearly see something in common: a historiographical blueprint, circulating among the members of a political elite, and in the end: in France, it led to a revolution which finally abolished it, or rather all the estates became one, that is, all citizens became equals; in Britain it survived as an institutional relic, which still can exert some political influence. In Soviet Russia, on the other hand, a new social estate was officially brought to the fore of historical processes: *the proletariat*. In ancient Rome the *proletarii*³⁸ were

[m]en without property. Originally the term was applied to persons not registered in the classes of the centuriate organisation³⁹ because they had not even the minimum property required for the lowest class. Their sole possession was their children, *proles*; hence the name. The *proletarii* were the poorest stratum of the population [44, p. 657].

The proletariat was resurrected after almost two millennia in the *Critique of Hegel's Philosophy of Right*, which is generally “regarded as a crucial text in Marx's intellectual development” [251, p. 105]. In it Marx attacks the Roman Catholic Church's three-estate model, and famously declares that

³⁸Latin plural of *proletarius*.

³⁹The *comitia centuriata* was a “popular assembly based on the division of the people into *centuriae*, classified according to the value of the property of the individual citizens” [44, p. 398].

[r]eligion is the sigh of the oppressed creature, the heart of a heartless world and the soul of soulless conditions. It is *the opium of the people* [309, p. 131, my emphasis].

Later on in the text, he announces his discovery of “the positive possibility of emancipation”

in the formation of a class⁴⁰ with radical chains, a class in civil society that is not of civil society, an estate⁴¹ that is the dissolution of all estates, [...] [in] a sphere, in short, that is the complete loss of humanity and can only redeem itself through the total redemption of humanity. This dissolution of society existing as a particular class is the *proletariat* [309, p. 140–141, my emphasis].

The intentional dissolution of society would correspond “to a political transition period in which can be nothing else but *the revolutionary dictatorship of the proletariat*” [311, p. 95, emphasis in the original]. The dictatorship of the proletariat entered a heated debate among the theoretical descendants of Marx, and finally it was picked up quite early by Lenin himself, the founder of the Soviet state:

The mere presentation of the question – “dictatorship of the party or dictatorship of the class;⁴² dictatorship (party) of the leaders, or dictatorship (party) of the masses? – testifies to most most incredibly and hopelessly muddled thinking [...] the Russian Bolshevik ... cannot help regarding all this talk about “from above” or “from below” ... as ridiculous and childish nonsense, something like discussing whether a man’s left leg or right arm is of greater use to him [279, p. 41, 49, emphasis in the original].

⁴⁰Marx uses the word *Klasse* in the original German text, which is considered the German lexical equivalent of the English word *class*.

⁴¹Marx uses *Stand* in the original German text, which is considered the German equivalent of the English *estate*.

⁴²In the Russian original Lenin uses the word *класс* [*klass*] which is considered the Russian equivalent of the English *class*.

We can clearly see that in Lenin's political theory there is no distinction between a social estate, a social class, or even a social caste. The only that matters is the proletariat which is basically identified with the communist party, and which is going to establish its own dictatorship towards a classless (or estateless) society. The one-party state was established after the October revolution, and the one-class, or rather no-class, society was proclaimed. The early Soviet regime became totalitarian, and controlled every aspect of social, intellectual and academic included, life. It reached its apex with Stalin, whose death in 1953 marked a major change in the Soviet regime. In 1961, during the Twenty Second Congress of the Communist Party of the USSR, with Khrushchev as General Secretary,

[t]here were several major ideological innovations brought out in the Programme [of the Communist Party]. The most significant was the abandonment of the dictatorship of the proletariat in favour of an all-people's state [439, p. 15].

All-people's state was the last and failed attempt to revive the original communist project: "[t]he widening gap between official rhetoric and ordinary lives . . . can ultimately be traced back to Khrushchev's failure" [p. 22] to deliver.

The defining characteristic of the proletariat, and in general of the social classes in society, according to the Marxist-Leninist sociological worldview, was labour, and the way the system of resources, means of production, and economic value were organised around it. They obviously ignored any concept of gift economy, as most of today's policy and decision makers do, as well; they considered as banking system only the financial capitalist banking system; and they scoffed at any other theoretical construct different from their own as "bourgeois", that is, as excessively middle-class and oriented to property and means-of-production relations. Among the first, in fact, to introduce the social-estate model into sociological literature and expand its anthropological semantics was Weber. According to Weber a social estate ⁴³

⁴³An alternative translation of Weber's *Stand* has been *status group*. Here the use of "estate" has been employed due to the grand scale of occupational groupings in the USSR.

“shall mean an effective claim to social esteem in terms of positive or negative privileges” [453, p. 305], and manifests itself is mainly through connubium, that is, low percentages of group intermarriages, commensality, that is, the people one eats with on a regular basis, and “monopolistic appropriation of privileged modes of acquisition or the abhorrence of certain kinds of acquisition” [p. 306]. In other words, membership in a social estate is mainly “revealed in consumption patterns, leisure activities, and friendship circles” [435, p. 4]. The first and foremost characteristic, though, of any social estate is its exclusive access to a number of certain resources and privileges.

Being a socialist state, private ownership of property in the USSR was prohibited, at least until Gorbachev’s reforms in 1986, and housing space was allocated by Soviet enterprises or by local councils. In the beginning of the 1980s “the public sector accounted for 77 percent of all urban accommodation” [8, p. 26]. Private housing, on the other hand, “[i]n Soviet terminology . . . belongs to individuals on the basis of their right to ‘personal’ property [*lichnaya sobstvenost*’], and is never referred to as ‘private’ property [*chastnaya sobstvenost*’]” [p. 26] which is supposed to be a means of production, and therefore a source of illegal (in socialist terms) income (see also [332]). There was also accommodation owned by trade unions and various co-operative organisations (for more see [8]). It has been widely acknowledged though that “Soviet citizens have suffered from a dreadful housing shortage; [. . . moreover] inequalities have existed in the distribution and consumption of accommodation” [9, p. 190]. Most well known forms of housing were the communal apartments (*kommunalki*), and

living [there] consisted of an apartment space shared by anywhere from four to seven families. Each family had its own room, which functioned simultaneously as living room, study, bedroom, and dining room. Shared with other families were the corridor, the kitchen, the bathroom, and the telephone. [31, p. 615].

“Status group” does not seem to connote this magnitude in scale in the same way as “social estate” does. See also Parsons’s comments on this [452, p. 347, n. 27; p. 424, n. 1], as well as [450].

If someone wanted to get their own flat, then they had to follow a certain time-consuming allocation procedure and any “attempt to improve one’s own housing situation [could end up] a full-time occupation” [332, p.239]. Even more difficult was to acquire a *dacha*, that is, a second country house. Academicians, on the contrary, that is, members of the Soviet Academy of Sciences, were among those “[p]articularly favoured by law, presumably because many work at home” [p.241]. One of the interviewees described how her father, a member of the Soviet Academy of Sciences, was given by the state a piece of land to build a his own “personal” dacha:

And there was an academicians’ settlement where only members of the Academy could get a piece of land. There were other academicians’ settlements, like Mozzhinka,⁴⁴ if you know. Before the war Stalin donated a piece of land with a building. But after the war the government gave the land to the Academy to build something by yourself if you want. And this settlement was built after WWII. And there were great people there. We had for example such settlements for theatre actors, for movie actors. We have still that dacha. There are not any more academicians there, you can buy land there now, after perestroika; it’s very expensive by the way. That settlement was founded, around 1957.[...] In my childhood I was acquainted with many scientists there, like Ginzburg was my neighbour, a Nobel laureate, and so on. There were physicists, mathematicians, and others.

A very important moment in all of the interviewees’ life was when they went to the mathematical secondary schools, a Soviet tradition established mainly by Kolmogorov. The pre-university school in the Soviet Union lasted initially ten, and later eleven years, with the last two spent in specialised schools for gifted pupils. The Kolmogorov Boarding School of the Lomonosov University was set up with the intention of inviting pupils from rural areas of the USSR and in 1993 it instructed “about 200 pupils in grades ten and eleven each year from Russia, Byelorussia, Tatarija, Baschkirija, Osetija/North Os-

⁴⁴A village outside Moscow.

etija, Checheno-Ingushetija and Kabardino-Balkarija” [93, p.110]. There were other schools specialising in mathematics for Moscovites, in Leningrad (today’s St. Petersburg), in Novosibirsk, as well as in other Russian cities and all capitals of the other Soviet republics. One interviewee recalled his years there:

School no. 63, as far as I can remember. [...] We had a very special atmosphere. We were all in love with science. We were all dreaming to become scientists. We had a special teacher in mathematics Sergei Nolaievich Voskresenski. He passed away though. He was very romantic about us. He was in love with science. He encouraged us. So at once I knew I must go to one of the highest universities, in Novosibirsk, and therefore, after classes I took homework [laughs], then, I remember, I made a schedule for myself at exactly 6.30, 18.30, I went to walk with my cousin, his name was Yuri, he passed away later. We went to walk for two hours. And at 8.30 I came home, and I studied mathematics with a special collection of problems in mathematics. It was very advanced at the time. Most of it was Modenov.⁴⁵ It was published as far as I remember in 1957, and it a collection of almost all problems which were given in Russian universities in entrance exams. [...] so I solved all of them, all of them. Therefore when the next day, whatever [they did] in mathematics, I was prepared much more than anyone else, as I was the first one who answered the questions Not because I knew what kind of questions there will be, but because I was trained.

We could easily say that this pupil was very diligent. But all the interviewees were in fact diligent pupils, as well as thousands of others not interviewed. We cannot therefore talk about pupils’ diligence on the grand scale, but that solving problems was actually a *consumption pattern* characteristic to the social estate of Soviet mathematicians, and physicists, up to a point. Solving problems could easily become a consumption pattern taking up a lot of one’s

⁴⁵Petr Sergeevich Modenov: his books with solved problems are still in print in Russia.

leisure time for a very simple reason: solving mathematical problems, is like playing golf; it is not necessary to have a great number of resources at one's disposal. Conducting a chemical, or even a nuclear, experiment, on the contrary, is a more resource-demanding consumption pattern, and because of that it does not become a consumption pattern, but remains in the sphere of work. Solving problems in fact becomes a *lifestyle*, and in the end the boundaries between work and leisure become rather blurred. In colloquial Russian there is actually a certain word describing the cultural stereotype of a pupil or student who is absent-minded and very absorbed in his or her academic thoughts: *botanik*, the original meaning of which is botanist.⁴⁶

In one of the interviews, the interviewee's eldest son was present, in order to assist in the interpretation from Russian into English. He is a mathematician himself and graduate student of the Lomonosov University, and he made occasionally some interventions in the interview:

Interviewee. We had very good life there [i.e. in the boarding school]. We had professor Dynkin, professor Gelfand, professor Smilga, if you know, and so on. And we had very good professors in literature. [...] Now there is a community in the whole world. Former students of the school.

Son. Actually I was recently in some gathering, maybe anniversary of that school. People from all over the world come to Moscow and they come from other countries once a year. Not many of them, but a lot of people come every five years, ten years, on anniversary. Great "crowd" of people.

Interviewee. It was a very interesting time in the Soviet Union. From one side there was no freedom, and on the other side, you had your own community, you have your kitchen to speak about these themes. There was a lot of pressure from the government. And as a result people got very close to each other, and very friendly to each other. It was Krushchev-Brezhnev period.

⁴⁶Compare with the English words *egghead*, *geek*, or *nerd*.

In general, informal networks in the Soviet Union were very important, a pattern which, as we can see from this interview excerpt, passed, and still passes, on to the younger generations. In general, though, the specialised high schools had made a general impact on the interviewee's lives in many respects. Older-generation interviewees, especially those who had finished school before 1953, had not attended specialised schools, because these became widespread later. Still, they went to *krushki*, that is, mathematical circles, which were extra-curricular activities on mathematics. The *krushki* were part of the Pioneers movement, the Soviet equivalent of the Western Scouting movement. There were also similar extra-curricular activities on chess, ballet, and others.

In a centrally organised state, although the elite “does its best” to control everything, the end result is increased reliance on informal networks: it is both impossible and very costly to try to control every detail of the state apparatus. Moreover the designers of state control might have missed some aspects of social behaviour. Informal connections might become then not only acceptable, as “legitimate initiatives”, but also necessary for the viability of the economic organisation of the system. There was, for example,

the informal role of the *tolkach* [“expediter”] in Soviet enterprises, who, though his position is never mentioned in enterprise statistics, serves the vital function of coordinating the material-technical supply in an inadequately functioning market-like distribution [435, p. 14].

A Soviet university was indeed an enterprise whose resources were mainly human resources. The expediter in that case was more institutionalised, but still he would take an “institutional initiative”, as we can clearly see from the narration of an interviewee:

It was in '52. And that time, after high school, I had a diploma with very good grades. And then people who had the diploma of this kind could avoid just entrance examinations. But they could have some kind of discussion with the examination committee. This committee gives them some problems, some general

questions, and all that. And I must say that my exam in this committee was very bad. I could not solve the problems which were given to me when I'm surrounded by three people. [...] So I was not accepted to the Moscow university. [...] My grandfather had many students. And when they learned that I was not accepted they went to see the rector of Moscow University, Petrovski, a very good mathematician, and a very nice person, and told him that this is really a very bad thing, what has happened to me. And Petrovski had some quota, some number of possibilities just to accept people from his list. So he just took me this way.

The rector, in other words, had the quota at his disposal, and, therefore, he could easily take "legal initiatives" according to his judgement.

These informal networks among Soviet mathematicians, and among Soviet fellow workers in general, were not only networks of acquaintanceship, as is usually the case in most Western European and North American universities. One interviewee explained how he was studying mathematics with other fellow students in the Lomonosov University:

We taught ourselves. For example I had friends who are a little bit older than me. Two years older, one year older. But because they were older they knew much more than me. And they taught us. I asked them a lot of questions. And at least two of them became really great mathematicians. [...] They are really great mathematicians, and they were great when they were students. They taught me a lot. And you know I can just take the receiver [right now], call, many different guys, and they will explain all questions I ask with great pleasure. They want nothing special on that. Ah, yes! Another one friend of mine during these years was Grisha Margulis. Eventually he got a Fields medal. And all them you can call them and ask them. It was very very usual thing. So for example, I can get a one-hour lecture by phoning them [laughs]. You know. So the whole situation was quite different.

And this is basically how we learned mathematics.

Many interviewees, when referring to other fellow Russian mathematicians with whom they had been together in the Soviet Union as fellow workers, mentioned them as their friends. In fact, “the Russian word *drug* has a stronger connotation [of intimacy] than the English *friend* and even stronger than the German *Freund*” [435, p. 94] In one interview, for example, during which the interviewee was narrating an event, at some point paused for a few seconds, and starting thinking by filling the pause with the conversation filler “er”: she was looking for an English word similar in meaning to the word “acquaintance”. My knowledge of Russian at the time was rather limited and I proposed the word “friend” in an interrogative grammatical mood, having in mind the idea of a “casual friend”, that is, an acquaintance whom one meets on a regular basis. She rejected immediately my proposal, and then elaborated as “someone I knew.”

When an interviewee was asked on his everyday life as an undergraduate student in MGU [Lomonosov University] his answer was as follows:

You know. People were staying there for a very long time. So They didn’t really have a need to leave the building. So they were walking, in this corridor in a circle, say two–three students, maybe with professors or without professors. They were all talking. And it was interesting that from the back of the elevator, you could see, from the conversation you could understand on which floor which people would get out. You know, physics, mathematics, you name it. [...] The seminars started always in the evening, because from all Moscow people could go to the MGU seminars.

What was happening with the universities, in other words, was that they were *institutions of socialisation and social integration* into the Soviet society. This is a conclusion that has been little referred to in the English-speaking Russian and formerly Soviet, studies. There has been excessive analysis on the political system, that is “from above,” and later on the everyday life of Soviet citizens, that is, “from below”. In fact in the Soviet Union, there seems

to have existed, actually, social estates, rather than social classes, or castes, or, needless to say, a homogeneous proletariat. And mathematicians seem to have been no exception to this. The concept of a class is basically related to income, which was in general not that important in the Soviet Union. So other social-economic phenomena seem to have made their appearance. The only Western scholar who supported this view, and inspired actually this section, was the German sociologist Teckenberg, who in 1981 declared that “Soviet enterprises are functionally equivalent to Western communities” [435, p. 15] (see also [436]); and while communities are characterised by *social inclusion*, enterprises are defined by *social exclusion*. A very similar approach to the estatist approach is Shlapentokh’s feudal model of modern Russia (see [413]).

The most prestigious award one could get as a Soviet mathematician, was to become *Doktor Nauk*, that is doctor of sciences. The first doctoral degree awarded in the Soviet Union, and is still awarded, was the *Kandidat Nauk*, that is, candidate of sciences. This was actually equivalent to the American or British Doctor of Philosophy (PhD), and was highly esteemed in the both in the USSR, and in the West. When a *Kandidat Nauk* was awarded by a university, it then had to be approved by the Higher Attestation Commission [*Visshaia Attestatsionnaia Komissia* (*VAK*)], an all-union governmental agency regulating the awarding of higher academic degrees and ranks, such as that of a professor. In Western Europe the award of a doctorate degree is conducted by the universities, without governmental intervention. In the Soviet Union this awarding had to be approved by this governmental agency, and usually the members of *VAK* were already recognised senior mathematicians, and members of the Academy of Sciences. Moreover “its membership had to be approved by the Central Committee [of the Communist Party]” [260, p. 39]. The really important academic degree, adding social standing within the academic world, though, was the *Doktor Nauk*, which was awarded either on the basis of a major scientific breakthrough, or on the grounds of the overall contribution to the field. The Soviet mentality as to the contribution on the field was eloquently described by an interviewee, working today in a North American university:

The scientific level was essential. But since the level was very high, it was difficult to reach this level. In other institutions a person could be considered extraordinary; at Landau [Institute] it would be considered good, not extraordinary... I would give you the example of Grisha Margulis. You probably heard the name: a Fields medalist. He got Fields medal, and didn't get a doctor degree. It was not considered education, the Fields medal, that somebody should be considered a doctor. No, not necessarily [laughs].

And when asked in his case, his answer was as follows:

In my case I never got it. I went to the West where it doesn't exist, in the USA, so I am not a doctor. No I am not a doctor. It doesn't matter for me now. But, in fact, if I go back to Russia, it will be important to be a doctor. I think I have enough [publications] to be a doctor [laughs].

A major contribution in the fields of differential geometry and ergodic theory was made by the Soviet mathematician Dmitri Anosov, which was published in English in 1969 [10]. He introduced in this publication what later came to be called *Anosov diffeomorphism*. Its subject had to do with dynamical systems, that is, systems whose behaviour evolves over time, like the flow of a liquid in a pipe, or the population of a species in an ecological niche. The dynamical systems in that book were actually more abstract, than those mentioned, and n -dimensional, that is their representation was done in n -coordinates; this is a very simplistic presentation, but it is almost impossible to analyse further without assuming that the reader is at least a post-graduate student in mathematics. This publication was actually Anosov's (second) doctoral dissertation, awarded to him by the Steklov Institute of mathematics in Moscow. The process of defending a candidate's dissertation was the same, and still is, as that of a doctor's dissertation in Soviet Russia. Every research organisation in the Soviet Union had a *scien-*

tific council [uchenyi soviet].⁴⁷ The scientific council was the equivalent of the Western academic senate. Moreover, the public defence of the dissertation had to be carried out in front of the scientific council. In Anosov's case the scientific council met on 4 November 1965 [16]. From the thirty members members of the Steklov scientific council, there were present at that day twenty-one: three academicians, that is full members of the Soviet Academy of Sciences, seven corresponding members of the Academy, nine doctors of science, and two candidates of science. On 4 June 1965 five months exactly before the dissertation defence, the research personnel of the Department of Differential Equations of the Steklov Institute had met, listened to Anosov's preliminary defence, and proposed Anosov's dissertation be publicly defended in Steklov Institute [16, p. 35–38]. The department's decision was based on the preliminary review of the dissertation by professor V.A. Rokhlin of the Leningrad.⁴⁸ The chair of the department's meeting, had been Pontryagin, Anosov's postgraduate supervisor.

Back to Anosov's public defence, the chair of the council's meeting is Vinogradov, the head of the Steklov Institute, and the secretary keeping the minutes is the candidate S.A. Telyakovski. Among the participant audience of mathematicians are well known personages in the mathematical world: Pyotr Sergeyevich Novikov, an academician very well known in group theory and father of the Fields medalist Sergei Petrovich Novikov; Alexander Osipovich Gelfond, a corresponding member of the Academy whose name bears an important theorem in number theory, and allegedly he was the Soviet navy's secret cryptographer during WWII; Igor Rostislavovich Shafarevich, a legend in algebraic geometry and a well known dissident in Soviet times; Sergey Mikhailovich Nikolsky, another legend in function theory, functional analysis, and partial differential equations, who kept giving lectures until the age of 92; Yuri Vasilyevich Prokhorov, a famous student of Kolmogorov, whose theorem on probability measures extended probability theory to abstract spaces.

⁴⁷Although there have been proposed reforms in the educational system of modern Russia, what is going to be mentioned in the text still holds.

⁴⁸Today St. Petersburg State University.

Vinogradov, the chair, announces the commencement of the meeting, and then he proclaims the three official opponents [*opponenty*] to the dissertation defence: Alexandr Kirilov, Ilia Pyatetskii-Shapiro, and Yakov Sinai. He then gives the floor to Telyakovskii, the secretary of the council. Teliakovski reads Anosov's application documents, and then gives back the floor to Vinogradov. Vinogradov asks if there are any questions, there is none, and then gives the floor to Anosov to report the results set forth in his dissertation. After Anosov's presentation, Andrey Andreyevich Markov, whose father proposed the well known Markov chains, asks a question, and Anosov answers. Then Pontryagin asks a question and Anosov answers. Pontryagin asks another set of four questions, and Anosov answers another four times. Pontryagin was Anosov's postgraduate supervisor and blind since the age of 14. Then the secretary receives the floor from the chair and reads aloud the report from the meeting of the Differential Equations Department of the Steklov Institute, as well as Rokhlin's preliminary review of Anosov's dissertation. Then the chair gives the floor successively to the official opponents of the defence, and each time each opponent finishes, asks Anosov if he would like to add something. Anosov has nothing to add to each opponent's remarks. During the third opponent's speech, Pontryagin asks him two questions, and the opponent answers them. Then the chair asks if someone wants to open a debate – nobody wants – and gives the floor to Anosov for some concluding remarks. Anosov then comments on some points that the opponents raised. The chair then announces the end of the defence. Then three members of the council are elected on the spot as a vote counting committee: Shafarevich, Kudryavtsev, and Prokhorov. Then an intermission is announced to vote in favour or against the awarding of the doctoral degree. A ballot is set up and everyone votes secretly, as to whether Anosov should become a doctor. At the end of the voting, the head of the voting committee announces the result: from the twenty-one present members of the scientific council, Anosov unanimously received twenty-one (secret) votes in favour. The chair of the meeting announces the official acceptance of the results. He then summarises the two final decisions of the meeting: the council considers Anosov's doctoral dissertation as fully publishable; the council should petition to the Higher attestation Commission for awarding Anosov with the *Doktor Nauk* degree.

The most usual waiting time mentioned for a doctor's degree the interviewees mentioned was around ten years. Some got it earlier, some others due to their Jewish origin had to wait for more rigorous results, and were simply stuck and had no good research ideas for some time, something like a "researcher's block," similar to writer's block. In one case, for example, an interviewee mentioned a time of ten years without a fruitful idea, until a major breakthrough in his research. Some others simply jumped onto another field, more interesting for them. In one case, in which the interviewee was of Jewish descent got his doctor's degree outside of the Russian Soviet republic:

An in about ten years or so the idea came that I have enough to defend the doctor thesis. Because in Russia they have two, two theses. [...] So after talking to Khasminski we decided that I could start writing another, that I had enough publications and stuff to write another [thesis]. And all that was already based on my second research. The first was more famous, but the second thesis was based on the second direction [of my research]. It was second order optimality, or second order optimal estimation. [...] And in 1986 I defended my doctor thesis but at that time ...there were a few places where a Jew could defend a doctor's thesis. And one of such places was Lithuanian, Vilnius University. So it was tradition that almost all Jews went there. And so every other meeting they had, was one their own, and one guest [laughs] ... They were sufficiently independent, and I think they kind of liked it that they could afford this kind of independence.

The doctor's degree was actually a prerequisite to become a member of the Academy of Sciences, and there were many privileges for the academicians, as we saw earlier. But in spite of the collapse of communism a doctor's degree is still a very valuable academic rank, as one of the interviewees mentioned, when asked if the doctor's degree had improved his financial situation:

Actually in my case it wasn't so. In the former Soviet Union this was extremely the case. For example, your salary could increase

twice. But in the former Soviet Union. Unfortunately in my case I obtained my doctor's degree exactly when Soviet Union collapsed. So in my case I didn't get this. But I still get an increase in salary. You must have a doctor's degree to enter some positions.

The doctor's degree actually freed the holder from many other problems that arose due to the communist regime: it did not provided only tenure and better salary, but also those talented started to attract many students. And as social mobility in the Soviet Union was very limited, every doctor could create his or her own research niche much easier within the system. There was, in other words, a very high level of intra-estate functional differentiation. One interviewee from a North American university, for example, mentioned that in the USA the research mentality is

that students have to see a bigger world. And it's good for them to switch between places, advisors. So if you are a good student it's better to go somewhere else. Say Harvard, Princeton. And when you get your PhD, probably you can come here as a postdoc. As a result the whole field of mathematics is more homogenised here in the States . . . but as a result there is very few of what can be called a school.

In the Lomonosov University, on the other hand,

there were the schools of Gelfand, of Novikov, of Arnold, of Sinai, because typically the best students in the university go to graduate school as aspirants [postgraduates], to write their PhD under the same advisor. And at the same time it creates a kind of pyramid because senior students teach the younger ones and that unified certain areas. [...] I was lucky that I was in this Novikov school. And because of such people like Novikov they attract the best students. It was very strong. So I went [to him] and defended my Phd.

The Soviet system of organising mathematics research, in other words, was very heavily relying on the charismatic authority of its senior research staff; it was, in other words, very close to the peripatetic school of Aristotle in Classical Greece. And one faculty could host many such leaders. Although the fieldwork conducted for this thesis centred for practical reasons, basically on the Lomonosov University and the Steklov Institute of Mathematics both of which are in Moscow, it seems actually that this model, due to the Soviet-Communist political and social context, was actually widespread all over the USSR. Moreover, what kept this scientific social estate alive, was not the resources it could pool: they were meager in any case. What kept alive its members, both new and old, was its socio-technical imaginary of a mathematical kind, as one interviewee recalled during his postgraduate years:

Like in a seminar, I mean, you became an important part of the seminar. People were listening to what you were saying, it was interesting for them to know your opinion.

One interviewee of the old generation when asked how he had developed his geometrical imagination he answered as follows:

When I was at school, seventh grade, 14 years old, I wanted to solve a stereo-metrical problem. I couldn't understand it completely, because I i didn't have any imagination. Science imagination. I asked my parents, to buy me constructor [construction set]. [With] this constructor I constructed the problem stereo-metric tasks and analysed this problem with real scheme [3D design]. After that during the summer, three months, I had completely improved my imagination so that I could solve any stereo-metric problem without any...with closed eyes. I calculated. After that I could write only result[s].

In other words, his visual imaginary had assimilated the technical artefact, and he ended up solving problems only by imagining their solutions, and then writing them down. The written solutions were *not* the real three-

dimensional solutions, because they were on a two-dimensional sheet of paper; they were imaginary three-dimensional solutions, but the technology implementing the solutions was two-dimensional. In another instance, another interviewee was asked how he could imagine Hilbert spaces and Hilbert-space geometry which is infinite-dimensional:

It comes when you start to explain to students the four dimensional, and then the five dimensional space and they have difficulty. When you tell them “look I mean that mathematicians feel so comfortable there, it’s like their home, it’s like their sleepers [sleeping berths].” It’s just time, I mean take time to adjust and just to feel comfortable. To get this experience. It’s also suddenly you can feel I can do anything I like. And if you start to live in Hilbert space suddenly you also feel that you are very comfortable. [...] It’s just accumulation of intuition, so you solve problems, you read some books, some theorems, and then it accumulates in you and then you feel comfortable. Just it gets inside you. And to say when it will happen I cannot but suddenly you feel very comfortable. It’s your good friend. Hilbert space is a good place for mathematicians to be.

Mathematicians are the only ones who can understand infinite-dimensional proofs, but they do not have infinite-dimensional construction sets to use; and if infinite-dimensional objects do indeed exist, as some quantum theories imply, the human biological system brain–eye simply cannot perceive these artefacts. All mathematicians have as a construction set is two-dimensional pieces of paper with two-dimensional drawings on them. Another interviewee, working on general topology, a branch where generalisations of spaces are studied, he attempted to describe his own perception of the mathematical objects in his field:

I don’t see these visually. ...I can’t describe how I see. The best time I have when I think about mathematics, this is when I go to bed, and when I cannot sleep. But you try to sleep. And

so I think about these things, and it's almost like a dream. I mean you are awake, you're not sleeping, you're not having a real dream, but you're in the middle somewhere. And I don't know, I see some things, but I can't describe what I see. Also it depends a lot on what kind of problem you are thinking. Say I spent several years, my kids were growing up then and they played with my dots with all these papers I had written, and all these papers were full of diagrams. I really had to draw diagrams to see some things, those aren't really geometrical objects.

And here comes the most interesting thing in the life of mathematicians, that is, geometric proof: they have a subjective vision of space, they write it down on paper, and then their fellow mathematicians assimilate the material-artefactual modality of the proof, and then they form their own subjective vision of space. So actually, the Soviet mathematicians were not only a social estate; they were a social estate of shamans for a very single reason: membership to the estate presupposed envisioning imaginary proofs, which nobody else could see, a socio-technical imaginary with exclusive visionary access. And probably this exclusive vision which was unseen to the political elite protected them from the Marxist-Leninist Moloch.

4.6 Modern City-State or Post-Modern Nation-State?

Gift economies, on the individual level, have many similarities with *prosocial behaviour*,⁴⁹ that is,

behavior which is

1. performed by a member of an organization,
2. directed toward an individual, group, or organization with

⁴⁹As opposed to *antisocial*, or *psychopathic*, behaviour (see, for example, [53]).

whom he or she interacts while carrying out his or her organizational role, and

3. performed with the intention of promoting the welfare of the individual, group, or organization toward which it is directed

[67, p. 711] (see also [208]).

A subgroup of prosocial behaviours is altruism, that is, “intrinsically motivated voluntary behavior intended to benefit another” [140, p. 647]. When, for instance, a parent in a context of very scarce resources gives to his or her child the remaining amount of food to eat and himself or herself starves to death, so that the child survives, is not only an instance of prosocial behaviour, that is, altruism, but also an instance of gift donation. It is believed that prosocial behaviours have biologically evolved

because they

1. increase individuals’ survival to reproductive age,
2. increase the reproductive capacity of the individual, and
3. increase either or both of these tendencies in other members of the species that likely carry the same genes

Inherent in this argument is that evolutionary forces favoring altruistic behaviors often come into conflict with those forces that favor behaviors maximizing the survival of the individual [140, p. 652].

Gift donation among infants in stress has long been considered as a sign of empathy-related behaviour, as we can see from the following quite early observation:

In the second incident, Michael, aged 15 months, and his friend Paul were fighting over a toy and Paul started to cry. Michael appeared disturbed and let go, but Paul still cried. Micheal paused, then brought his *teddy bear* to Paul but to no avail. Michael

paused again, and then finally succeeded in stopping Paul's crying by fetching Paul's *security blanket* from an adjoining room [218, p. 612, italics added].

On the face of it we could say that Michael attempted to appease Paul's distress by "bribing" him with his teddy bear; Paul declined the "bribe" and stuck to his own security blanket. We can clearly see the importance of materiality in early childhood, i.e. the teddy bear and the security blanket, as well as gift donation as an empathically informed attempted appeasement, or "positive manipulation", of Paul's emotional distress. We can also see that children can learn gift donation as early as they can start speaking a language. The author of this thesis, in other words, believes that gift-donating behaviour is hardwired on the brain in a similar way as language is hardwired, and, moreover, gift-giving activities are socially regulated, as well, in the same way as language is socially regulated.

Today, an example of social regulation of gift-giving behaviour is the modern *open source* software movement, that is online communities of programmers who collaborate in software projects voluntarily and contribute as unsalaried workers. Linux, most probably, is the most well known such open source software project (see, for example, [451]). Each end user can download for free Linux, install it in his or her computer, modify its operations, and then redistribute the modified version free of charge. *Closed source* software, on the contrary, can be downloaded as *binary code* only, that is, in code readable only by computer, and not as *source code*, that is, in code readable by programmers. In this way closed source software, can be sold, but it cannot be modified and then resold. Although closed source software can be *non-rivalrous in consumption*, it can easily become *rivalrous*, that is, in "consumption of a unit of the good by one person [that] precludes consumption of that same unit by another person" [273, p. 155]. A case in point is *Adobe Photoshop*, owned and distributed by Apple Inc.: it is a graphics editor that gives a great edge to a professional photographer over his or her competitors who do not use it, because it can manipulate digital pictures and produce a multitude of new graphical products; when more and more professional photographers, though, in the same pool of potential customers start using

it, the competitiveness of each photographer who owns it diminishes until the group of professional photographers is saturated with the same software; then closed source software becomes nonrivalrous. Still, though, possession and use of a new and more updated version of Adobe Photostop can, again, give a great edge to the owner, until, again, the pool of professionals becomes saturated. The definition of a rival good, in other words, is not always useful for economic goods, such as software, as well scientific knowledge, the scarcity of which cannot be always taken for granted. Open source software programs, on the contrary, are modifiable by programmers who can read and understand the source code, and as such they have been characterised as *antirivalrous goods*, that is “the value of a piece of [open source] software to any user increases as more people use the software on their machines and in their particular settings” [454, p. 154]. And here is a great similarity between the open source movement and mathematics: the producers of the antirivalrous goods are, at the same time, its end users; mathematicians prove theorems for other mathematicians, not for the general public. The general public, though, is always free to copy and read theorems, but in practice, as with open source programming, reading a theorem of today’s mathematics is close to impossible due to the extremely high sophistication of modern mathematics.

Under normal circumstances the open source movement would not have survived in a modern free market economy, which places excessive importance on corporate profits, and therefore on closed source software. Richard Stallman, the mastermind programmer behind the open source initiative, proposed the *copyleft* licence with four essential freedoms, to “protect” open source software from copyright attempts [454, p. 48]:

1. Freedom to run the program for any purpose
2. Freedom to study how the program works and to modify it to suit your needs
3. Freedom to redistribute copies, either gratis or for a monetary fee
4. Freedom to change and improve the program and to redistribute mod-

ified versions of the program to the public so others can benefit from your improvements

He, then, called his copyleft copyright licence *General Public License* (GPL) and added one extra requirement that came to be called the *viral clause*:

[i]t is not permitted under the GPL to combine a free program with a nonfree program *unless* the entire combination is then released as free software under the GPL [454, p. 48–49, italics in the original].⁵⁰

It very easy to see that, in general, the General Public License has held, in fact, for mathematics, as well, for centuries; that happened, though, more due to *scientific custom*, rather than to (modern Western) copyright law. Any mathematician is free to use a theorem for their own purposes, as the first Freedom prescribes; most of what has been called “applied mathematics” is, in fact, implementation of the first Freedom. Any mathematician is free also to study how any theorem runs and to modify it to suit his or her needs. Bernhard Riemann, actually implemented the second and fourth Freedoms: he redefined the former definitions of straight line, named his new definition *geodesic* and unified theorems in both Euclidean and non-Euclidean geometries. Albert Einstein in his turn, implementing the first Freedom, adopted Riemann’s definition of geodesic, and, postulated, based on experimental data of his time, that light travels along a geodesic; the theory of general relativity was then born. The third freedom pertains, in fact, to the publication of scientific articles and monographs.

In spite of the “openness” in open source software, though, as well as in mathematical theorems, understanding Linux, or a complicated theorem in mathematics, such as Poincaré’s Conjecture, can be extremely difficult for a lay person, most probably impossible: these are for professionals to be understood with extended and persistent training. Even mathematicians themselves cannot understand theories outside their fields, as one interviewee admitted. Still, though, in a gift economy, such as the economy of Soviet

⁵⁰For more on open source software and public licences see [7, 43].

mathematics, a gift could be donated only by a craftsperson of the trade, and not by an outsider; a gift, in other words, was, and still is, material evidence of *contributory expertise*, which, according to Harry Collins,

enables those who have acquired it to *contribute* to the domain to which the expertise pertains: contributory experts have the ability to *do* things within the domain of expertise.” [102, p. 24, italics in the original].⁵¹

If we turn to the training of social scientists to see how contributory expertise can be achieved there, then, according to [243], teaching approaches of research methods in the social sciences can be grouped roughly in three categories: methods that make social research visible; methods promoting learning by doing research; and methods focusing on the research process. Active learning, in particular, that is, “any teaching method that gets students actively involved” [242, p. 35], as opposed to the lecture mode of teaching, is one method of making research visible to the trainee, the purpose of which is to prompt the undergraduate student to engage more actively with a certain topic during classroom instruction. One parameter, though, that seems to be missing here is that classroom instruction is, in fact, active learning in “blackboard disciplines”, such as mathematics, philosophy, and the older discipline of philology. Solving mathematical problems, in particular, *is active learning in mathematics* during which chalk and blackboard have been simply replaced by pencil and paper. When later on, the experimental sciences of physics, chemistry, and biology came to the scientific fore, active learning extended from writing to engaging in the laboratory mode of instruction.

Still though, contributory, as well as interactional, expertise on its own is not enough to organise a scientific community in the grand scale, such as that of Soviet mathematics. If, for a moment, we consider Soviet mathematics as a scientific cult in a grand scale, then we can glean some interesting similarities to the sacrificial model of early ancient Greek religion:

[T]he act of killing and eating [the sacrificed animal] is in itself

⁵¹See also [130, p. 101–121] and [102, p. 24–27]

transient. It may, despite the emotion of the dramatic killing and of the collective participation in the meal, leave no physical trace of itself. The consequent vital need for continuity [along institutional spacetime] may be expressed in three ways:

- (a) in the continuity of *deity*, who remains to demand the same form of sacrifice at regular intervals or in specific circumstances;
- (b) in the *subjective* continuity of the human insistence that in the future the animal be killed and distributed in the same way;
- (c) in continuity of an established *object or place* (altar, shrine, temple), in which the preservation of the durable part of the animal, or of the implements of sacrifice, may be a first step [404, p. 48–49, italics in the original].

The *deity* behind the workings of scientific activity is *Truth*: a disembodied, and depersonalised deity, that is, a universal signifier mobilising, even in a perfunctory fashion, the scientific community. The *subjective* continuity of the human insistence is the imaginary resource of *systemic trust* in the scientific social system as a human society *always* in transition, *always* in search of truth. The *objects or places*, whose materiality preserves the social system over institutional spacetime, are (mainly) the scientific articles and monographs, as well as the university buildings where research takes place. Moreover, if we consider classes, lectures, research meetings, oral defences, and so on, as the repeated patterns of everyday scientific activity, than we can propose that, scientific activity involves *ritualization*, that is,

the strategic production of expedient schemes that structure an environment in such a way that the environment appears to be the source of the schemes and their values [40, p. 140].

The involvement of ritualized behaviour embedded in everyday activities, in other words, transforms, indeed, the mathematical community into a *scientific cult*.

The materiality of the scientific article does not reduce the importance of informal discussions among scientists, or the struggle among political factions

within scientific communities. These, though, are not the concern of this thesis, since they have always been considered as implied and their existence has always been accepted as self-evident among social researchers in STS. What seems to have been neglected more, though, in the STS community is the scientific article as the *stabilising factor* of the scientific communities, rather than the destabilising processes of informal discussions and political factions. A project of such a grand scale, such as the Soviet mathematical enterprise, could not reach so gigantic proportions without strongly stabilising elements within its institutional structure. The insistence on the monetary aspects of Soviet mathematics illustrates the importance of systemic trust in the Soviet scientific social system as a strong stability element. The most important aspect in the coinage of money, dating back to ancient Greece was the involvement of the city-state, or rather a state authority, as *the* guarantor of *materially produced*⁵² trust:

[t]he *state* may be involved in issuing money, controlling it, guaranteeing it, enforcing its acceptability, and so on. For instance the state may decide to guarantee the weight and purity of pieces of metal by stamping them. Or it may decide to so guarantee merely the value. In other words, the state may either exclude the possibility of disparity between substance and appearance or make it irrelevant [404, p. 19].

In the Soviet system the guarantor of (materially produced) trust was(were) the editor(s) of the scientific journal, since there were no anonymous peer-review processes involved. At the same time, the scientific article never reached a level of *fiduciarity* proper, that is, an “excess of the fixed conventional [i.e. nominal] value of pieces of money over their intrinsic [i.e. commodity] value”. The scientific article remained attached to its authors, and as such, its commodity value remained attached to its authors/private-issuers as well as to its journal/institutional-issuer.

⁵²“Trust produced” not only as in “products produced” but also as in “documents produced”.

The materiality of the article, that is, the material container of the imaginary commodity of mathematical intelligence, made also *hoarding* possible. One of the interviewees mentioned that Alexander Gelfond,⁵³ known from the *Gelfond–Schneider theorem* in number theory, was the chief cryptanalyst of the Soviet Navy during WWII with the rank of a general. Although this information could not be confirmed by an independent source during the author’s fieldwork in Russia, that could indeed be possible. Number theory, as well as its adjacent field of commutative algebra, is known to be used in cryptography and cryptanalysis (see for example [250]); as it was written back in 1987,

it is no longer inconceivable (though it hasn’t [officially] happened yet) that the N.S.A. (the agency for U.S. government work on cryptography) will demand prior review and clearance before publication of theoretical research papers on certain types of number theory [250, p. v].⁵⁴

An article, though, written by two of his students and celebrating Gelfond’s sixtieth birthday anniversary mentions that “[d]uring the war years [i.e. WWII] Gel’fond [sic] served in the VMF” [*Voyenno-Morskoy Flot (SSSR)* = Military Maritime Fleet (of the USSR)] [371, p.234]. A more interesting aspect of the Soviet scientific social system was that its systemic trust extended well over its borders: shortly after the success of the Sputnik satellite, in a Cold War political climate, many research projects on the Soviet educational system were launched by American universities and institutes, and many English translations of Russian books and monographs as well as of Soviet scientific journals flooded the North American universities; Russian was now becoming one of the compulsory language choices for North American PhD students in both mathematics and physics along with German and French which were the traditionally required languages for PhD research in the natural sciences (see [231]).

⁵³Not Gelfand, but Gelfond.

⁵⁴Actually such an incident did happen in 1991 with Phil Zimmerman’s cryptographic program *PGP* (see [283, p.187–225]).

The material container of a scientific article or a monograph points to a fundamental property of materiality, that is, to the property of *visibility*: a word has to be *heard*, an article has to be *read*, and, in general, the imaginary aspect of any material artefact has to be *induced* by its material container, that is its *material trace*. The material trace of an article is the way the article's physical properties enter human consciousness, and as such it is important for integrating its value in the community. The visible aspect of an article, the aspect that a university library lends, or the aspect that a mathematician reads on a computer screen, *is not* the article itself, but its material trace which can be carried along by any human. The imaginary object, though, corresponding to its material container, cannot be "grasped" with the hand any more. This misunderstanding has led to a confusion as to what is an exchange economy, and what is circulation of an economic commodity. Although not everybody in a particular field can understand what exactly an article is about, the details of a proof or a theorem, everybody can understand what a scientific article is, and what its social function is, that is, to *integrate its social value into the community*. We saw earlier that in gift exchange "[d]isplay and exhibition as realized in exchange [...] verify the resources of giver and receiver: objects and object qualities become definitely associated with particular persons" [153, p.24]. If defining then gift exchange, in other words, as the integration of a gift to the community, an article as a gift is a *generalised medium of exchange* acting as "a structured expectation as well as a [material] symbolic mode of communication to others and to the [social] actor himself" [351, p.312], that is, its social semantics is simple enough to be understood by every member of the community; at the same time its social semantics points to its author(s) as concrete persons within the community. As for the *circulation* of an article, its material container, that is, the paper pages as well as the printed words and symbols on them, have to *circulate*: while exchange has to do with *authorship* and a *community of authors*, circulation pertains to *utility* and a *community of users*.

Before going further into the social semantics of a scientific article, though, it would be useful to briefly flash back once more to early ancient Greece.

Religious sacrifice in Greece involved burnt up animals (holocaust), or, more usually, commensal sacrifice which involved the cooking over fire and then sharing of the meat among the participants (see [123]). It is believed now that the first artefacts used as coins in Homeric Greece were iron spits that had been used for ritual sacrifice:

The characteristics that qualify iron spits to perform money functions are *portability*, *countability*, *durability*, economic *value* that is neither too great nor too small, *standardisation* of shape and size, *mass production*, the kind of familiarity that creates *communal confidence*, and *substitutability* for other objects [404, p. 104, italics in the original].

We can see that, apart from substitutability, scientific articles qualify for all the other properties. Another important development leading to the invention of ancient Greek coinage were the cylinder seals which were being used in ancient Near East the

common purpose [of which] [...] seems to have been to signal to any viewer that a certain person as an individual or member of a group was *present* at a certain act, be it as witness, as overseer, or as controller [343, p. 19, italics added].

Later on, in ancient Greece coinage, though,

[w]hereas seal-marks seem to embody the power of the owner of the seal, coin-marks create no imagined attachment between the coins and their source. [...] [C]oin-marks, unlike seal-marks, relate to the *material* on which they are impressed (metal, not clay). They authenticate the metal as possessing a certain value. And they do so not by transmitting power (magical or otherwise) to the piece of metal, but by imposing on it a form that recognisably assigns it to a distinct category of things, the category of *authentic* coins [404, p. 119, italics added].

The ancient Greek city-state, in other words, used metal sealing technology in the capacity of *guarantor* of coin authenticity, and as such, of *controller* (as well as *comptroller*) of money circulation. Ancient Greek coin marking, in other words, is very reminiscent of modern *commercial branding* practices; the branding, though, conducted by a scientific journal, is that of *scientific validity*.

Coming back to authorship and utility, we can now say that the *inalienability* of a scientific article under the capacity of a donated gift means that during its circulation its authors remain the same, but its users can constantly change. A user, though, can become an author by modifying knowledge in older scientific articles, and exchanging his or her knowledge with the community, that is, putting new knowledge into circulation. Turning our attention, for a while, to modern usual coins and banknotes, we can see that there is no individual author on it: the only “seal” present is that of the corresponding central bank issuing the banknotes. Since we have many scientific journals issuing articles on different subjects, we can say that in science, in general, as well as in Soviet mathematics, in particular, we have a *free banking system*, rather than a central banking one. Moreover, the central bank in a financial social system, as well as the scientific journal in a scientific social system, are the *authors of systemic trust*. In a financial social system a central bank, is the *single* author of systemic trust, while in the scientific social system there is *double authorship*: the scientific journal, as an institutional author of general scientific validity, as well as the individual author(s) of each particular article as authors of *specialised* scientific validity.

The main institution in the USSR behind the “branding” of science was the Academy of Sciences of the USSR, apart from the Ministry of Education, uniting the scientific lives of its members and employees as an overarching super-institution:

The Soviet Academy of Sciences ... was the place of employment of the most outstanding fundamental researchers in the country. These scientists spent their lives in its service, and their places of residence, travel and vacation privileges, and health and social

services were traditionally controlled by it [182, p. 2].

Most of the mathematical journals were published by the *Nauka*⁵⁵ Publishing House, called *Publishing House of the USSR Academy of Sciences* before 1963. Moreover, the chief editors, and quite often the majority of the assisting editors, the *collegiate of editors*, were academicians, that is, members of the Academy of Sciences. Future mathematicians, though, were funnelled quite early in their careers into the mathematics departments through special high schools with a curriculum quite enhanced in mathematics. Most of the prestigious mathematical faculties, though, such as those of Lomonosov University, for example, Leningrad State University,⁵⁶ or Novosibirsk State University, were extremely demanding for students who had not attended special schools; one interviewee, for example, mentioned that some of his fellow students in Lomonosov University who had not attended classes in mathematical schools as pupils simply dropped their studies, or were registered to less prestigious mathematical departments. And after five years of study, not four as in most Western European countries,⁵⁷ a student who wanted to pursue an academic research career as a professional mathematician, was obliged to have by the end of the final year in the university published already two articles. Then to become a postgraduate research student had to pass three exams: one in a foreign language important for his or her field, that is, English, French, or German, for mathematics; one exam in his or her scientific field; and one exam in the philosophy of Marxism-Leninism.⁵⁸

American universities, on the other hand, are organised in a different way. Pupils, after finishing high school, apply for admission to a college which awards an *associate degree*, if it is a two-year college, or a *bachelor's degree*, if it is a four-year college. Initially, in colonial America, colleges were independent institutions offering general education curricula to pupils of differing

⁵⁵That is, science, in Russian.

⁵⁶Today Saint Petersburg State University.

⁵⁷Communist Poland, until very recently, had also five years of compulsory university attendance, the final year being an integrated Master's degree.

⁵⁸Today these entrance exams are still the same with the exception of Marxism-Leninism: students today are examined in the history of (Western, Russian, and Eastern) philosophy.

educational backgrounds:

[b]ecause of the backwardness of isolated rural schools and the absence of national standards of primary and secondary education, students with widely differing academic skills found themselves entering colleges at the same time, but unable to cope with the work [444, p. 253].

As a result the first two years in a college were devoted in homogenising the educational background of the students. After the devastating American civil war, though, a merging process between colleges and universities began:

[i]t proved more economical for colleges to merge administrative facilities with nearby medical schools, law schools and other schools of graduate studies. Although some colleges continue to function independently, most American colleges are now part of larger, all-encompassing universities [444, p. 253].

The first two years of attendance to an American college there is no specialisation offered; on the contrary students are offered a curriculum of general education. In four-year colleges, during the last two years of attendance the student chooses a primary specialisation, a *major*, and quite often a secondary specialisation, a *minor*:

Designed to encourage students to explore the limits of knowledge in a particular subject, the *major* often requires studies in one or more related fields. Such secondary studies are, at some colleges, called a *minor* and usually involve far fewer courses than the major [444, p. 673, italics added].

After college a student with academic aspirations enrolls to a university as a postgraduate research student. Postgraduate studies, for a student of mathematics, last about five years. A few of the interviewees employed by American universities mentioned the main difference between the Soviet and the

American university in undergraduate majoring: in the Soviet Union mathematicians were specialising very early in their careers, while in the US they specialise quite late. Still, the American universities kept US on the edge of science and technology for many decades during the Cold War. After, though, the disintegration of the Soviet Union

over 1,000 Soviet mathematicians migrated to other countries, with a large fraction settling in the United States. [...] [Moreover,] the American mathematicians whose research programs most overlapped with that of the Soviets experienced a reduction in productivity after the entry of Soviet émigrés into the U.S. mathematics market. [...] [There is also] evidence that the students of the Soviet émigrés had higher lifetime productivity than other students from the same institution who had nonémigré advisors [58, p. 1146].

When John Charles Fields, a Canadian mathematician, proposed an award for major accomplishments in mathematics research back in 1932, he

proposed to found two gold medals to be awarded at successive International Mathematical Congresses for outstanding achievements in mathematics. Because of the multiplicity of the branches of mathematics and taking into account the fact that the interval of such Congresses is four years, it is felt that at least two medals should be available. [278, p. 62].

These medals, later on, became four in number, were later named after Fields, and acquired a rather legendary and controversial status in the international mathematical community (see [28]). It is indeed interesting to note a rather informal indicator, not very rigorous in mathematical terms, but still quite illuminating as to the impact of the Soviet scientific social system in mathematics research: since 1990 until 2010 there has always been a Soviet mathematician awarded a Fields medal with the exception of 2006, during which there were two former Soviet mathematicians. Every four years

a Fields medal awarded is not a collective accomplishment to be discarded light-heartedly.

Chapter 5

Material Theorems - Phantasmatic Proofs

5.1 Introduction

Mathematics as a science has changed tremendously during the 19th and 20th centuries. After Descartes' innovation of coupling algebra and geometry into, what is called today, analytic geometry, four fundamental major changes of tremendous importance occurred, without, of course, downgrading other equally important innovations. The first, during especially the 19th century, was what has come to be called *differential geometry*: the theory of curves and surfaces was generalised to n dimensions, called now *differentiable manifolds*, and the concept of *curvature* was introduced. This was Einstein's theoretical base in his mathematical formulation of the theory of general relativity, where spacetime, as a 4-dimensional manifold, that is a generalised surface, can be curved in the presence of vast concentrations of mass. The second major innovation has come to be called *algebraic geometry*, and has gone unnoticed by the lay and the general sociological public. It took place during the 50s and 60s and its major innovation was to expand, like in differential geometry, the definition of curves and surfaces, now called *algebraic varieties*. The defining property was to connect a generalised curve or surface to the zeros, i.e. the solutions, of polynomial equations on n variables, by considering them as a set of numbers with the locally defined operations of addition and multiplication, i.e. as an algebraic ring or field, to use the current mathematical terminology. The overall concept of a mathematical space, as a generalisation of the usual natural space of everyday living, changed for a second time. The third major change of a mathematical space, which has also gone unnoticed, has taken place in the last 20 years, and has come to be called *noncommutative geometry*. Explaining the change in simple terms is impossible, for the very simple reason that this was actually a fusion of differential and algebraic geometry (initially this theoretical venture was called *algebraic differential geometry*) under the influence of the fourth fundamental major innovation in mathematics, that is *functional analysis*, which is going to be the object of the present chapter. The idea behind functional analysis was to create an *infinite-dimensional Euclidean geometry*, and modern probability theory, for example, is actually this: a version of an infinite-dimensional calculus, if calculus is defined as differentiation and

integration techniques on a Euclidean n -dimensional, that is finite, geometric space. These four fundamental changes were of course gradual, and not explosive, and each one of them happened within a generation. They led to a multitude of major new mathematical innovations, some of which are very popular today in the insurance and finance industries: these of probability theory, mathematical statistics and stochastic analysis.

The general public, including sociologists of science, are, in general, unaware of these developments. There is a general perception that modern mathematics have to do with numbers, geometry, or proof. But if one tries to prove a theorem in an infinite-dimensional geometry to a lay person he or she will get deeply confused. What could be the point of a proof, in the end of the day, if the intended audience of that proof cannot simply comprehend it? Yet, mathematicians have proved theorems which are beyond the common comprehension. This chapter, as a text presenting specialised knowledge, could have three kinds of potential audiences: a lay audience, an early postgraduate in mathematics audience, and an audience of professional mathematicians. An audience of professional mathematicians would spend most probably no more than five minutes in reading and understanding it. The geometrical figures presented to demonstrate and strengthen a mathematical fact, would be practically useless: professionals have assimilated the textual artefacts and they can simply think the proof. The early postgraduates would most probably be at a mild loss; they would try to copy and complete the proofs they do not understand and it would take them a bit of a time to read it. The act of writing down the proof in fact has the reverse effect of gesturing: instead of a hand following the directives of a mind in its cinematic flux of consciousness, now the mind restructures its cinematic consciousness following gesture; and accompanying figures now help more in that.

A lay person would most probably be at a loss: symbols would simply be impenetrable, figures understandable only as geometrical paintings without any deeper meaning, and in general the whole text, apart from the scattered English words and phrases, everything would “be like Greek” to him or her; assuming, of course, that this fictional lay person is literate. A lay per-

son would understand nothing about the “proofs”. We have in other words, three kinds of audience: an audience of professionals, that is, the fully blown shamans who have no problem in “seeing” the spirits; we have the postgraduates, that is, the apprentice shamans, who are still on probation “trying to see” the spirits; and finally we have the outsiders, the “infidels”, who will “never see” the spirit-world of mathematicians and they will have to rely on the mathematicians’ word. This is a good illustration of the idea of a *scientific cult*, that is, a community of shamans. Some of the developments, though, of modern mathematics, in the author’s opinion, have to be presented to a general public, and then let the public decide for themselves: do they want to expend the extra effort necessary to understand it or not? But the general conclusion in the sociology of science that comes out of this presentation of professional mathematical proof is of fundamental importance: is it possible any more, or in the future, to pursue a sociology of science by someone who is not versed in that particular arcane discipline? This question becomes more pressing when the field under research is quantum physics, financial mathematics, or algorithmic financial trading, three highly technical fields. For the time being, though, there seems to be no such a problem.

5.2 The Artefactuality of the Book

Most of the mathematical books of the 50s, 60s and 70s were hardbacks, for reasons of endurance: they would be publicly available, and lots of pairs of hands would open and touch them; many thumbs, index and middle fingers would turn the pages, with or without saliva; sometimes they would fall on the floor and forced open. There had to be a technology of resistance. Such were most of the mathematical books in the Soviet Union. So the first qualities that someone would notice in a hardback book immediately are its weight and hardness. If a toddler is given a book what would he do with that? Lift it and experience its heaviness, throw it and test his or her abilities in throwing, smell it to see its particular scent, maybe try to bite it to see how it tastes. But a book once thrown up or ahead opens easily, its pages are being browsed open during its short flight. So our toddler would be happy

and curious to discover that a hardback can become soft once opened. It could be said right now that the experience of the materiality of the book is still not institutionally constructed. The toddler experiences its materiality through his or her senses and various sensory organs scattered around the body, such as the gravity sensors in the joints of the arms and hands. But even grown-ups have these experiences of a book. A quite common habit, for example, of book lovers is to smell the pages of books, especially old ones. During a scene in a recent popular blockbuster movie, *The Bourne Ultimatum*, taking place in Algiers, the protagonist, a former CIA super-spy, in his effort to defend himself in a small room against a hitman who is holding a knife, grabs a big hardcover book which he uses as a shield to defend himself and then as a weapon of attack. The whole scene is so fast pacing that a lay person wonders if it is possible to think about such defensive properties of a hardback in a few split seconds. Or in another popular movie, *The Day after Tomorrow*, the son of the protagonist and his friends, survive a deep freeze hurricane in New York in a public library, locked in a room, gathered around a fireplace and burning books, in order to warm up. In both these cases, as well as in the case of the toddler, we see that the imagined and experienced properties of a hardback book are not dictated by social convention but by the resourcefulness and innovation of the *users*. But we already know that this is not what a book is for.

So what is a book for? We are going now to try to (re)discover the (institutional) purpose of a book, a certain book, in particular, directly related to our study. Necessary prerequisites for studying mathematics in the Soviet Union as an undergraduate were *elementary set theory, topology, and functional analysis*. We will see later each one of them more specifically, after we have developed the necessary phenomenological framework. All these are contained in Kolmogorov and Fomin's book on functional analysis [254] which is still popular for the training of postgraduate students. We are going to base our observations on its English translation [253], popular, as well, in English-speaking universities. Besides the titles, the main difference between these two is that the Russian edition has some extra chapters, which we considered as undergraduate prerequisites in the Soviet universities .

Also the Russian edition is more condensed in its presentation, quite usual in the Russian tradition of mathematical tradition of book writing, while the English one is rather detailed in some topics, in simplified way, written in a more pedagogic fashion. A very interesting observation is that the Russian edition is interested in developing mathematical concepts, and extending the student's "mathematical maturity". The concept of *concept* is directly related to cave art painting. We will see later how. Let us see now the first paragraph of the first chapter of the Russian book; the reader is asked to intently study, and then try to copy it by hand on a white sheet of paper ([254, p.13], , italics in the original have been replaced by typewriter font):

В математике встречаются самые разнообразные множества.
Можно говорить о множестве граней многогранника, точек на прямой, множестве натуральных чисел и т.д. Понятие множества настолько общее, что трудно дать ему какое-либо определение, которое не сводилось бы просто к замене слова «множество» его синонимами: совокупность, собрание элементов и т.д.

To those who do not know Russian, or simply cannot read aloud Cyrillic, that is indeed a great challenge. Even writing can be problematic, especially the letter "ж". If we try the same with italics the situation even seems worse and confusing:

*В математике встречаются самые разнообразные множества.
Можно говорить о множестве граней многогранника, точек на прямой, множестве натуральных чисел и т.д. Понятие множества настолько общее, что трудно дать ему какое-либо определение, которое не сводилось бы просто к замене слова «множество» его синонимами: совокупность, собрание элементов и т.д.*

We can see now how the materiality of the form is extremely important. Due to years of prolonged schooling, and in the case of highly educated people excessive schooling, one tends to forget the tremendous effort exerted by

primary school pupils to learn how to read and write; one tends to forget as well that a toddler learns sound language, the material form of which is sound: toddlers have no idea of grammar, words, verbs, prepositions, object, indirect subject, or, in the case of Russian, perfective and imperfective verbs or genitive and dative cases. We can try to transliterate the above text into Latin characters

V matematike vstrechajutsja samye raznoobraznye mnozhestva. Mozhno govorit' o mnozhestve granejmnogogranika, toчек na prijamoj, mnozhestve natural'nyh chisel i t.d. Ponjatie mnozhestva nastol'ko obshhee, chto trudno dat' emu kakoe-libo opredelenie, kotoroe he svodilos' by prosto k zamene slova "mnozhestvo" ego sinonimami: sovokupnost', sobranie elementov i t.d.

Now the material form has become more familiar, we can read and hand-write the text, understand the meaning of some words which seem familiar, but still, we can not infer its meaning, if we do not have a certain knowledge of Russian. Let us now try to translate it into English, using the Oxford Russian dictionary:

One can find in mathematics the most diverse of **sets**. One can talk about the set of [geometric] faces, the points on a straight line, the set of natural numbers and so on. The concept of a set is so general, that it is difficult to give any definition for it, which would not be reduced simply to the substitution of the word "set" by its synonyms: aggregate, collection of elements and so on.

Now there is no problem in understanding the text. Its materiality is a familiar one, although there might still be some problems as to what is a face, or a point. The reader could also wonder whether natural numbers are numbers living free in the wild. In other words, while the text now is readable and understandable, there are still some problems in understanding it. Let us now cite another mathematical excerpt formulating the so-called *Kolmogorov's extension theorem* ([347, p. 11]):

For all $t_1, \dots, t_k \in T, k \in \mathbb{N}$, let ν_{t_1, \dots, t_k} be probability measures on \mathbb{R}^{nk} such that

$$\nu_{t_{\sigma(1)}, \dots, t_{\sigma(k)}}(F_1 \times \dots \times F_k) = \nu_{t_1, \dots, t_k}(F_{\sigma^{-1}(1)} \times \dots \times F_{\sigma^{-1}(k)})$$

for all permutations σ on $\{1, 2, \dots, k\}$ and

$$\nu_{t_1, \dots, t_k}(F_1 \times \dots \times F_k) = \nu_{t_1, \dots, t_k, t_{k+1}, \dots, t_{k+m}}(F_1 \times \dots \times F_k \times \mathbb{R}^n \times \dots \times \mathbb{R}^n)$$

for all $m \in \mathbb{N}$, where (of course) the set on the right-hand side has a total of $k + m$ factors.

Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stochastic process $\{X_t\}_{t \in T}$ on Ω , $X_t : \Omega \rightarrow \mathbb{R}^n$, such that

$$\nu_{t_1, \dots, t_k}(F_1 \times \dots \times F_k) = P[X_{t_1} \in F_1, \dots, X_{t_k} \in F_k]$$

for all $t_i \in T, k \in \mathbb{N}$ and all Borel sets F_i .

Now we are faced with more problems than the translated text: we can read and write the text, we know the words, we understand the subscripts and superscripts, but still to the uninitiated into mathematics, it makes no sense; it still has no meaning. And in the end one could wonder what is meaning in mathematics?

Summarising the above observations, when we are confronted with an open book “the first thing we notice is the visual perception of signs” ([224], p.19). These signs can be handwritten, or printed. Moreover,

in fluent, fast reading we do not perceive the individual letters themselves, although they do not disappear from our consciousness... The first process of reading a literary work [as well as a mathematical one] is thus not a simple and purely sensory perception but goes beyond such a perception by concentrating *attention* on the typical features in the *physical* or *phonetic* form of the words” ([224], p.20, my emphasis).

This attention to the material form of the text can be called as the *coupling of mind with matter*. It can be attention to the thickness or the colour of

the fonts, or if it is a serif or sans serif typeface. In the case of martial arts, for example, as we saw in the beginning of the section, this kind of attention turns to the felt weight, or hardness of the book, since it is going to be used as a weapon of defence or attack; whether the book is written in Russian, English, or ancient Greek, is totally irrelevant to the contextual circumstances of a street fight. Put simply, the institutional context (including the shortly lived institution of a street fight or a chance encounter) dictates the collective perception (or rather intentionality) of a corresponding assemblage of artefacts, and not the artefacts themselves. Since

the meaning of a word can be considered in two different ways: as part of a sentence or a higher semantic unit, or as an isolated single word, taken by itself” (p.24),

we can see that the meaning is itself a (perishable) assemblage of sound artefacts, whether attention is on the pronunciation of each individual phoneme, as in begin-ning to learn a foreign language, or on each word, as in being taught the grammar of a language, or on a whole sentence, as in an everyday speech. Also the pronunciation of a word or a phoneme can be the topic of a joke, or an attempt to demonstrate higher status. In each social context therefore the context defines the assemblage, on which attention is drawn. Moreover, the material of the assemblage of artefacts is not always sound; it can be material space, as is the case of a room together with the position of personal belongings, combined with the coordinates the user of the room has to orientate himself or herself; the more messy a room, the more demanding the personal coordinate system, obviously. The assemblages here are actually distances among objects, concentrations of objects in various points in the room, objects as coordination points or difficulty in finding an object. The materiality, in other words, of a room with personal belongings can be divided by the person using the room, into various sub-materialities, which in return can be divided into smaller ones, or can be combined into larger sub-materialities, attaching at the same time spatial meaning into each one, creating a *spatial encyclopedia* of interconnected meanings. This mental ability has been called *spatial cognition* ([341]). In other words there is no human

cognition without artefacts, and no artefacts without human cognition.

5.3 Counting Infinity

Set theory is the first theory that unified mathematics. It created the common conceptual framework, which made possible communication among mathematicians of different and very diverse fields. Its originator was Georg Cantor, who became famous (and infamous) in the mathematical circles of the 19th for his definition of infinity and infinite sets. This was a major change in the mathematical perception of infinity([431]). This could be described as a Kuhnian paradigm shift: the resistance it met as a mathematical theory has become legendary ([111], see esp. p.1, 266). Today there are no such disputes over set theory, and there are no disputes among mathematicians in general, since proofs *based on* set theory have eliminated scientific consensus problems. Behind set theory, actually, was lying the problem of foundations of mathematics: what is proof, what exists in mathematics, mathematical ontology and so on. The end result of this process was the establishment of the *axiomatic method*: the mathematical community *accepts* the existence of certain fundamental mathematical entities and the validity of certain propositions without proof (*postulates* in Euclid's language, the originator of this scientific world-view); moreover, it is accepted that some fundamental concepts are *primitive*, that is they cannot be defined. The most famous axiom, for example, was Euclid's 5th postulate:

if a point does not belong to a straight line, both lying on the plane, then only one other straight line can be drawn from that point, which is parallel to the original one.

This proposition, although obvious, remained, and still is, unprovable within Euclidean plane geometry. It can be proved only when the algebraic methods of analytic geometry are introduced; then the straight lines can be formulated as solutions of first degree polynomials of two variables on the two-dimensional analytic plane, the \mathbb{R}^2 , as it is traditionally denoted in black-

board bold typeface. But in the formalism of mathematics \mathbb{R}^2 is not the same as the Euclidean plane, since on the Euclidean plane only the ruler and compasses are allowed to be used for geometrical constructions: a practical engineer's demand, in fact. What interests us here is that the community of mathematicians reached a consensus regarding set theory. This consensus, moreover was achieved by taking into consideration additional proposals as to how to eliminate controversies and contradictions, arising in various problems to be solved; it was not achieved by itself, as the general reader might be led to believe. It was achieved by the *creative proposals* of some scientists, who managed to *convince* the community as to their usefulness in everyday problem-solving practices.

Set theory interests us here, because of its presence throughout all of mathematics. It is a very fundamental subject in university mathematics departments, and all the undergraduates have a rough knowledge of it. Actually it is the basic common communication conceptual framework between working mathematicians, as regards their professional everyday research life. The first chapter of Kolmogorov and Fomin's book is devoted to set theory. Cantor's definition of a set, which is still being used in set theory textbooks, was:

By an 'aggregate' [i.e. a set] we are to understand any collection into a whole M of definite and separate objects m of our intuition or our thought ([79], p.85).

This definition was rejected as a mathematical definition, actually it is evidently not, so a set was, later considered as an (undefinable) *primitive notion*, i.e. it was left to the reader, or the researcher, to get a conceptual grasp of it. Mathematicians learn how to handle the concept of a set during their undergraduate years; it is, in other words, part of their scientific socialisation processes. The first chapter, therefore, in Kolmogorov and Fomin's book is mainly a revision, rather than a main part of the book. A set is usually denoted by capital letters of the Roman alphabet (in the Russian text as well), e.g. A or B , type-faced in italics. Their *elements* (or *points*) are denoted by small Roman letters, e.g. x or y , type-faced, as well, in italics. The *membership property* is denoted as $x \in A$ (read *x belongs to/ is a member*

of/lies in A) . The concepts of a set, an element, or membership are primitive, so no formal definition is given. On the contrary, many examples of sets and their elements are given in the classroom, or in a textbook. Underlying the cited examples is the assumption that actually we are mainly interested in sets of numbers. If we know that the elements of a set are a, b, c , and d , we can denote that as $A = \{a, b, c, d\}$. We are faced now with a very well known problem in philosophy: *intentionality*; where are these mathematical symbols and concepts pointing to? Materiality means the ability to point to a material location or property, such as a place, a colour, a sound, and so on, which is publicly perceived by any other human in the vicinity. So a set becomes *visible* by *writing down* on a paper, by *type-facing* on the pages of a book, or by *talking about* it. At the same time it is a concept, i.e. it is *imagined*. These fundamental mathematical concepts, therefore, have *both material and phantasmatic support*. This constitutes the *public sphere* of mathematics: visible materiality and shared fantasy. Every mathematician has a personal and intimate perception of mathematics, but, at the same time, they can understand one another's perceptions by materially formulating their proposals, maybe in an informal way, as well, even though their common fantasy space may be totally inaccessible to the non expert.

If an element x does not belong to a set A , this is denoted by (or type-set as) $x \notin A$. And here is the first point of one truth proposal, in order to avoid the contradictions inherent in Cantor's theory: *if A and B are sets, then $A \notin B$, and $B \notin A$* ; in other words a set cannot belong to any set, itself included. A set whose elements are sets is called a *collection*, *class*, or a *family*, of sets. Although a collection is handled in the same way as a set in computations, they are ontologically different in the *fictional universe* of set theory: collections are "big" sets. Collections are usually denoted by capital letters of the Roman alphabet and type-set in Kuenstler script typeface: \mathcal{A} , \mathcal{B} or \mathcal{C} ; in the Russian text they are type-set in Fraktur typeface: \mathfrak{A} , \mathfrak{B} or \mathfrak{C} . Families are usually collections of *indexed sets*: $\{A_i\}_{i \in I}$, where I is the *index set*. In stochastic calculus, for example, *families of collections* with the set of *nonnegative numbers* as the index set are in use: $\{\mathcal{F}_t\}_{t \geq 0}$, or, in the Russian text, $\{\mathfrak{F}_t\}_{t \geq 0}$. Letting aside now the formalities of mathematical

logic, and assuming by belief that what we are going to define does exist, as most of the working mathematicians do, we are going to define *operations* on sets. The *union* of two sets A and B , denoted by $A \cup B$, is a new set, let us say C , whose elements are the elements either of A or B : $C = A \cup B$. The *intersection* of A and B , denoted by $A \cap B$ is a new set, let us say D , whose elements are the common elements of A and B : $D = A \cap B$. If A and

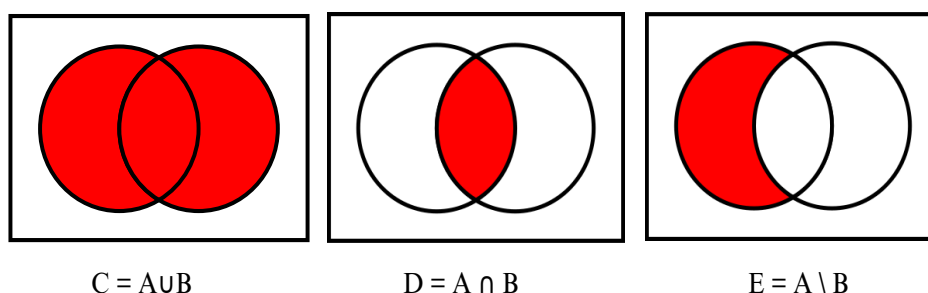


Figure 5.1: Venn Diagrams

B have no common elements, then their intersection is the *empty* set, that is the set with no elements, denoted by $\{\}$, or \emptyset , which is a capital letter borrowed from the Danish and Norwegian alphabets. Note that $\{\emptyset\}$ is not the empty set, but a set whose only element is the empty set: $\{\emptyset\} \neq \emptyset$. The *relative complement* of B in A , denoted in the past by $A - B$, nowadays by $A \setminus B$, is a set E , whose elements are the elements of A which do not belong to B : $E = A \setminus B$. The reader should bear in mind that so far no specific reference has been made as to the particular elements of any set; as long as their material (visual in particular) denotations exist, that is not necessary. The postgraduates do not need this reference, the undergraduates have already enough experience in previous mathematical subjects in the course of their studies prior to choosing a set theory course. There are also unions and intersections of families or collections of sets: $\cup_{i \in I} A_i$, and $\cap_{i \in I} A_i$. Quite often *Venn diagrams* are used to illustrate graphically the operations between sets (Fig. 5.1). Finally, when all the elements of a set A belong to a set B , then this is denoted by either $A \subset B$, or $A \subseteq B$ (read A is a subset of B), or by $B \supset A$, or $B \supseteq A$ (read B is a superset of A). In the Russian text only the first ($A \subset B$) and the third ($B \supset A$) denotations are being used.

Before proceeding to a consideration of *infinity* in mathematics, we have to examine the concept of a *function*. Functions, in early undergraduate courses, are actually (deterministic) computational mathematical formulas producing numbers:

$$f(x) = x^2, \quad g(x, y) = \frac{\sqrt[3]{y^{\frac{3}{2}+x} - 7 + 5 \cos y}}{x + \tan(\frac{y^3}{2}) + 3}, \quad h(z) = \frac{\arg z + e^{i\pi z} + \bar{z}}{|z|^2 + 3}$$

In more advanced courses of calculus and mathematical analysis, functions are perceived as *mappings*, that is as mechanisms associating the elements of one set, the *domain*, with those of another one, the *range*. The function $f(x) = x^2$, for example, associates numbers with their squares: it *maps* the set of (the usual) numbers \mathbb{R} *into* itself; this is denoted by $f : \mathbb{R} \rightarrow \mathbb{R}$. The function $g(x, y) = x^2 + iy^2$, where i is the imaginary unit, maps the Cartesian plane \mathbb{R}^2 into the complex plane \mathbb{C} , that is $g : \mathbb{R}^2 \rightarrow \mathbb{C}$. This conceptual change in the perception of a function has a very important computational consequence: it is not necessary any more to know the explicit formula of a function, as long as we know some of its properties; and this is the situation in real life problems: deriving some properties from experimentation and educated guesses, the mathematician tries to approximate the function under investigation, and, if possible guess its formula. This is going to be important in stochastic calculus, because predictions are based on functions of unknown formulas but of known particular properties. Another, more interesting at the moment, example is to consider the set of *natural numbers*, $\mathbb{N} = \{1, 2, 3, \dots\}$, the set of *even numbers*, $2\mathbb{N} = \{2, 4, 6, \dots\}$, and define the function $f(n) = 2n, n \in \mathbb{N}$. We can clearly see that f maps \mathbb{N} into $2\mathbb{N}$. If, reversely, we define the function $g(m) = \frac{m}{2}, m \in 2\mathbb{N}$, then g maps $2\mathbb{N}$ into \mathbb{N} . We can therefore associate every element of \mathbb{N} , with every element of $2\mathbb{N}$, in a reversible way, which, additionally is *one-to-one*: to every element of \mathbb{N} corresponds only one element of $2\mathbb{N}$, and vice versa. The same methodology can be observed by a shepherd who wants to check if there are any stray sheep: having a number of pebbles equal to the number of his sheep, he drops one in a small bag, every time one sheep enters the pen; when a sheep is missing, then a pebble will remain “orphan” on his hand. In the case of counting even or natural numbers we saw that when we followed the same procedure as

the shepherd and use pebbles from the “bags” of natural or even numbers correspondingly, we are left with no “orphan pebbles” on our hands. What Cantor did, and annoyed many of his contemporaries, was to assert that, since there is a *one-to-one* function from the set of natural numbers \mathbb{N} *onto* the set of even numbers $2\mathbb{N}$, that is into the whole of $2\mathbb{N}$, the sets \mathbb{N} and $2\mathbb{N}$ have “the same amount, or multitude, or quantity”, of elements, they are of the same “size”. He called this “infinite amount of elements” *cardinality*, and he denoted the cardinality of the set of natural numbers by \aleph_0 (read *aleph-naught*, *aleph-null* or *aleph-zero*), which is the first letter aleph of the modern Israeli Hebrew alphabet, always typeset in serif typeface. The zero subscript denotes that \mathbb{N} is the “smallest” infinite set. Cardinalities in finite sets are the usual numbers: the sets

$$A = \{1, 2, 3, 7, 0\}, \text{ and } B = \left\{ \sqrt{5}, \sin \frac{3\pi}{5}, e^{\frac{\pi i}{7}}, \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}, \log_2 37 \right\}$$

have the same cardinality: $\text{card } A = \text{card } B = 5$. We already saw that $\text{card } 2\mathbb{N} = \text{card } \mathbb{N} = \aleph_0$. Sets whose cardinality is finite or \aleph_0 are called *countable* (or sometimes *denumerable*) in mathematical analysis, *discrete* in probability theory, or *digital* in telecommunications. Sets whose cardinality is greater than \aleph_0 are called *uncountable*.

We have given a definition of a function, of infinity, we can handle numbers, but still there is no material support of these besides the typeset visual representations of these; we still can not answer to a toddler where he or she can find some numbers in the wild; even in the case of shepherd, the pebbles are still pebbles, and not numbers. Like Shakespeare’s King Lear, or like the recently filmed Batman and Bane of DC Comics, numbers, sets, collections and cardinalities are the *acting characters* of the *fictional universe* of mathematics: they *animate* mathematical narrative.

Since $2\mathbb{N}$ is countable, it can be written as an indexed set with \mathbb{N} as the index set: $2\mathbb{N} = \{a_n\}_{n \in \mathbb{N}}$; or, alternatively, $2\mathbb{N} = \{a_n | n \in \mathbb{N}\}$, where the vertical dash “|” is read as “with the property.” The indexed set $\{a_i\}_{i \in I}$ is called a *sequence* if the index set I is the set of natural numbers \mathbb{N} . Countable sets, in other words, can be written as sequences. If a set A is finite, $A = \{a_1, a_2, \dots, a_k\}$, then it can be written, as well, as a sequence by setting $a_n =$

a_k , for all $n \geq k$. The set of *integers* $\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$, that is the set of positive and negative natural numbers is countable, $\text{card } \mathbb{Z} = \aleph_0$. The set of *rational numbers* $\mathbb{Q} = \{\frac{\kappa}{\lambda} \mid \kappa \in \mathbb{Z}, \lambda \in \mathbb{N}\}$, that is the set of positive and negative numbers which can be written as fractions, the finite and periodic decimals included, is countable, as well: $\text{card } \mathbb{Q} = \aleph_0$. All these, of course, and what follows are being proved in Kolmogorov and Fomin's book, and also in any standard textbook of set theory. So the question arises: are there any examples of uncountable sets? The answer is positive. The set of *real numbers* \mathbb{R} , that is the set of all known and still unknown numbers, is uncountable, $\text{card } \mathbb{R} \succ \aleph_0$: it cannot be written as a sequence. A second question arises. The *powerset* of a set A , denoted usually in advanced texts by 2^A , is the collection of subsets of A : $2^A = \{B \mid B \subset A\}$. Then the powerset of a set is proved to be of higher cardinality than the original one, that is $\text{card } A \prec \text{card } 2^A$. Since $\mathbb{N} \prec 2^{\mathbb{N}}$, and $\mathbb{N} \prec \mathbb{R}$, is there any possibility that $2^{\mathbb{N}} \simeq \mathbb{R}$, that is $\text{card } 2^{\mathbb{N}} = \text{card } \mathbb{R}$? This used to be called the *continuum hypothesis*. In 1966 it was proved by Cohen that this is actually an axiom: *the existence of an infinity bigger than the infinity of the natural numbers and smaller than the infinity of the real numbers cannot be proved or disproved within the formal system of modern mathematics*. Put simply, it is a matter of belief and faith, rather than science. In the historical course of mathematics, the assumption that the proposition $2^{\mathbb{N}} \simeq \mathbb{R}$ is true proved very useful, and mathematics since then is based on that. Since the time of the proof of the independence of the continuum hypothesis, a set A whose cardinality is the cardinality of \mathbb{R} , $\text{card } \mathbb{R} = \aleph_1$ (read *aleph one*), has been traditionally called *uncountable* in mathematical analysis, *continuous* in probability theory, and *analog* in telecommunications. There are also other interesting topics of set theory included in Kolmogorov and Fomin's book, which are not going to concern us since they go beyond the scope of this thesis.

Let us now try to follow very carefully the proof that the *closed unit interval* $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ ¹ is uncountable, as presented in the book of Kolmogorov and Fomin. We are not interested so much in the proof

¹If the reader feels overwhelmed and confused by the many formulas and definitions introduced in a small amount of space, this has been done intentionally.

itself, but rather on the imaginative prerequisites to understand the proof. The first thing to bear in mind is that we are talking about infinity, that is something that has no ending in real life; the reader, therefore, of the proof has already a conscious or unconscious idea of what infinite means. For the undergraduate student of mathematics, since he or she is going to be examined, it is imperative to have an idea of this; during the examinations a student has to coordinate his or her eye with the hand, in order to write, and this can be done only through his or her vision of the infinity, among others. A toddler, in a similar fashion, coordinates his or her hands and eyes in order to build imagined constructions with Lego bricks; when two toddlers play with the same Lego bricks, then we are witnessing a conversation between them, that is, in the current sociological parlance, a *Lego focus group*.

Our concern now is if the set $[0, 1]$ can be written as a sequence, that is in the form $[0, 1] = \{\alpha_n\}_{n \in \mathbb{N}}$, with the condition that if $\kappa \neq \lambda$, then $\alpha_\kappa \neq \alpha_\lambda$ and vice versa, if $\alpha_\kappa \neq \alpha_\lambda$, then $\kappa \neq \lambda$. Since for any $n \in \mathbb{N}$, we have that $0 \leq \alpha_n \leq 1$ (we can formulate now normal sentences with the heavy use of mathematical symbols), every $\alpha_n, n \in \mathbb{N}$, can be written as a decimal number: $\alpha_n = 0.a_{n,1}a_{n,2} \dots a_{n,m} \dots$, where $a_{n,m}$ ($m \in \mathbb{N}$), is the m -th digit in the decimal expansion of α_n . First prerequisite for this proof, therefore, is the ability to abstract: the symbol α_n is the generic number which we do not know (and if we knew we would still not know, since a number does not exist materially) of the interval $[0, 1]$ which does not, as well, exist materially besides as a typeface on the book pages, or in handwritten form. What exists actually in real life applications are measurements producing numbers, but not numbers themselves. On the other hand the generic number α_n does exist, since an undergraduate can fail the examinations, if he or she considers this number as a figment of the professor's imagination; in other words the assumption that numbers *do* exist does have a real impact in social life, as everybody knows. The decimal expansion of a number is already known from school:

$$\begin{array}{lll} 1 = 0.9999\dots, & \sqrt{2} - 1 = 0.4142\dots, & \pi - 3 = 0.1415\dots, \\ \frac{1}{3} = 0.3333\dots, & 0.5 = 0.5000\dots, & \dots \text{and so on.} \end{array}$$

When reading the proof, consciousness zooms in on the writing symbols. So in order to facilitate the argument of the proof, the terms of the sequence together with their decimal representations are being graphically rearranged on the white page background in the following way:

At the same time it is always a good idea to write down this rearrangement, so that the reader can facilitate himself or herself by using the hands equipped with the pencil, or ballpoint, in creating new rearrangements. We see that still the watching eye and the writing hand must still be coordinated, since the view of the proof until it is understood is a *parallax view*. The fact that the view of the proof changes, and the hand, in concert with the eye, rearrange the symbols of the proof, indicate that actually consciousness is *cinematic*. In other words the symbols of the proof are being rearranged by means of the eye and the hand is being coordinated by this *imaginary footage* of consciousness in use. The researcher cannot possibly have access to this cinematic consciousness, but he or she can record the material rearrangements of the particular assemblages being used in a particular material context. This can be done, for example, in the classroom, or, especially in our case, during an informal coffee meeting between researchers. Returning back to the proof, observing the above arrangement, we can see that the decimal digits $a_{1,1}, a_{2,2} \dots a_{n,n} \dots$ form an imaginary infinite diagonal; this is denoted also by the fact that the double subscripts are the same. The proof is actually based on the handling of these subscripts, and on the ability to

communicate the validity of this particular handling to the audience of the proof.

Let us now choose the decimal $\beta = 0.b_1b_2 \dots b_n \dots$, with digits (from 0 to 9) $b_n \neq a_{n,n}$, for all $n \in \mathbb{N}$. In other words $b_1 \neq a_{1,1}, b_2 \neq a_{2,2}, \dots, b_n \neq a_{n,n}, \dots$ and so on. Then we can observe that the decimal β is different from the generic term α_n of our sequence, at least on the decimal digit of order n , that is $b_n \neq a_{n,n}$. We reach, therefore, the conclusion that actually $\beta \notin \{\alpha_n\}_{n \in \mathbb{N}}$, since, for all $n \in \mathbb{N}$, $\beta \neq \alpha_n$. Now we have just reached a contradiction, not a real conclusion, since we assumed that the unit interval $[0, 1]$ is countable. The new conclusion we reach is that our initial hypothesis was false, since every step in the proof was in concert with the laws of mathematical logic. The negation of the initial hypothesis, therefore, is true: that the unit interval is not a countable set. Since it is infinite, it is uncountable. We have to modify slightly the above proof in order to make it complete, but this does not alter the basic argumentation of the proof, which is (Cantor's) *diagonal argument*.

This section has introduced many symbols, many new definitions in a seemingly short amount of space. This has been done with certain purposes in mind. When the undergraduate is introduced into new subjects there are more symbols and definitions to remember, and we can say that actually the mathematical fictional universe is indeed overcrowded: the student himself or herself is going to impose his or her fictional order, in the end of the day, on this universe. The confusion that the non initiated reader might have felt was actually intentional. In the practice of mathematics exercises and problems play a very important role. The first role is to train the cinematic consciousness to coordinate the writing hand with the watching eyes. This is accomplished, for example, in the case of chemistry with the laboratory, or in mechanical engineering by assembling and disassembling a car by hand; this is what is actually meant by *hands-on experience*. This is the main obstacle, as well, in invention and innovation. A second very important role is to start forgetting the use of symbols and definitions, and begin focusing on new mathematical constructions, which will be forgotten again and the focus of *intentional attention* will be on the next constructions dictated by the

curriculum. The situation in the social sciences is totally different: the focus is mostly on reading texts, than writing. The only comparable situation in the social sciences and humanities happens when a new foreign language is learned, especially one with an alphabet different from the Latin one. Then the learner is under pressure to compile sounds, words, sentences and text on his or her own, or, in a more sociological terminology, to compile his or her own assemblages. When the primary school pupils learn how to read and write, the corresponding assemblages of sound, paper, pencil, desk, black-board, and so on, become *part and parcel* of their cinematic consciousness. Later in secondary school they *will have forgotten* that they use them, and their consciousness will be *withdrawing* to other levels of focus. Cinematic consciousness by engaging itself as the *who*, that is as *Homo faber*, with the *what*, that is the tool, the spoken word, or the written letter, cannot distinguish any more the *who* from the *what*: it has assimilated both of them in its *running footage* [421, see p.239–276].

5.4 Keeping Distances

The previous section was written with two purposes in mind: the first was to introduce the mathematical perception of infinity by trying to construct it materially and then imagine it, and then to give an idea of how different kinds of infinities are distinguished. It should be borne in mind that in proving that the set of numbers between zero and one, the numbers zero and one included,² Cantor’s diagonal argument does not construct the uncountable infinity of this set, as it is the case of the natural and even numbers; this is actually *proof by contradiction*. It is not a constructive or computational proof. Leaving aside the formalities of mathematical logic, we can see now that Cantor’s proof leaves the perception of uncountable infinity to the reader of the proof, as it claims only existence, but exposes no demonstration or a concrete way of understanding of uncountable infinity. This is actually a first serious indication of a very strong imagination behind proving or disproving as a mathematical practice, not only that of the author of the proof, as it is

²That is the *closed unit interval* $[0, 1] = \{x \in \mathbb{R} | 0 \leq x \leq 1\}$.

rather obvious, but also that imagination of the reader. as well: *a proof is not a proof, unless it can be communicated as such* (see [118]).

On the other hand too much reliance on imagination can lead to false conclusions; this can be avoided by peer reviews. When Andrew Wiles claimed in 1993 to have solved Fermat's last theorem, notorious in mathematical circles for its relentless resistance to any proof whatsoever, he created a sensation; it was discovered by peer referees after a few months that there was a major gap in the proof, which Wiles had missed and undermined seriously its validity. Later in 1994, he managed to solve that gap, and proved a theorem unproven for centuries [416, p.277–302]. The second, and more important purpose of writing the previous section is to illustrate the different academic cultures with respect to undergraduate studying between the social and the mathematical (and engineering, as well) sciences: while the future social scientist focuses on understanding the material by *reading*, the mathematician focuses on understanding the material by *writing*.³ In other words the usual participant observation is not enough in the conventional sense. In a usual anthropological study by participant observation understanding by writing is being accomplished by taking part into conversations: the *writing hand* in that case is the *talking mouth*. But this kind of “writing” is something that the social scientist has been trained for since very early in his or her career. But becoming an engineer, or a mathematician, or any profession that demands extensive and exhausting training in creating a multitude of artefacts and mechanisms, is not something the social scientist has been trained for. And the artefacts involved in cultures of low level material complexity can be used, in most of the cases, by simple imitation: there is a difference between using a sword to kill and using a nuclear missile to kill: they are both socio-technical systems, but there is a huge difference in the material complexity between them. There is a great gap in the sociological literature regarding a *phenomenology of high-complexity material culture*, something like Simondon's *technical individuation* [415], or his concept of *mechanology* (see [414]).

³This has been called the (*phono*)*logocentrism* of the social sciences [121, p.81-87], [421, p.136].

We have seen how complex has been so far the definition of a set. The sets and their points though are not enough to study space; we need more. We⁴ are going to enter now the concept of space in mathematics by defining the *metric*, or *distance*, function:

Definition 1. Let X be a set. A *metric function* ρ on X , $\rho : X \times X \rightarrow \mathbb{R}_+$,⁵ is defined as follows:

$$\rho(x, y) \geq 0 \quad (\text{non-negativity}) \quad (5.1)$$

$$\rho(x, y) = 0 \quad \text{if and only if} \quad x = y \quad (\text{coincidence axiom}) \quad (5.2)$$

$$\rho(x, y) = \rho(y, x) \quad (\text{symmetry}) \quad (5.3)$$

$$\rho(x, y) \leq \rho(x, z) + \rho(z, y) \quad (\text{triangle inequality}) \quad (5.4)$$

where x, y, z are any points of X . A set X **equipped with** a metric (function) is called a **metric space**, and is denoted by (X, ρ) .

The property of non-negativity, that is property 5.1, denotes something we would expect from a distance function: the distance between two points in space can be only a positive number or zero. Moreover we would expect that when two points coincide, then the distance between them should be zero, and vice versa, which is property 5.2. Property 5.3 denotes that the distance of x from y is the same as the distance of y from x (the property of symmetry). So far the properties defined are expected, or “common sense,” ones. The triangle inequality, property 5.4, is actually a well known theorem from Euclidean geometry (Fig. 5.2): “the length of each side of a triangle is always smaller than or equal to the sum of the lengths of the other two”; Euclides is still pulling the strings from the background.

We are going to see some examples of metric spaces now. It is important to note at this point that the definition of a metric does not contain its calculation, only the properties of it, after a particular measurement has been done. Which method of *measuring distances* is going to be applied is

⁴We the mathematicians, or we the outsiders observing the mathematicians.

⁵ \mathbb{R}_+ denotes the (set of) non-negative numbers and $X \times X$ denotes the *Cartesian product* of X with itself, that is the set whose elements are (*ordered*) *pairs of points*.

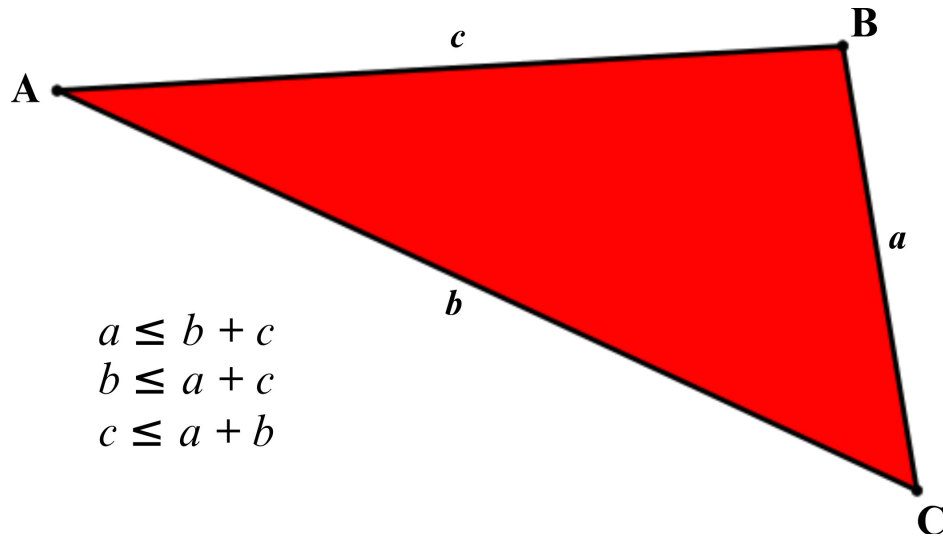


Figure 5.2: Triangle Inequality

left to the discretion of the reader, or the mathematician in general. Let us now see⁶ some more concrete examples of metric spaces.

Example 1. The set $\mathbb{R}^n = \{(x_1, \dots, x_n) | x_i \in \mathbb{R}, i = 1, \dots, n\}$ of *ordered n -tuples*⁷ becomes a metric space with the *Euclidean metric*:

$$\rho_{Euclides}(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ any points of \mathbb{R}^n . The metric space (\mathbb{R}^n, ρ) is called the *n -dimensional Euclidean space*, and x_1, \dots, x_n are the *Cartesian coordinates* of x .

If we observe the calculation of the Euclidean metric in example 1, we see that it is the square root of a sum of squares, that is the sum of the differences of the corresponding coordinates of the points. This is actually the Pythagorean theorem, which the definition demands to hold, when three points in \mathbb{R}^n form a right-angled triangle, that is a triangle whose one angle

⁶See as mathematicians or “see” as outside observers of mathematicians.

⁷This is actually a finite Cartesian product: $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}}_{n \text{ terms}}$.

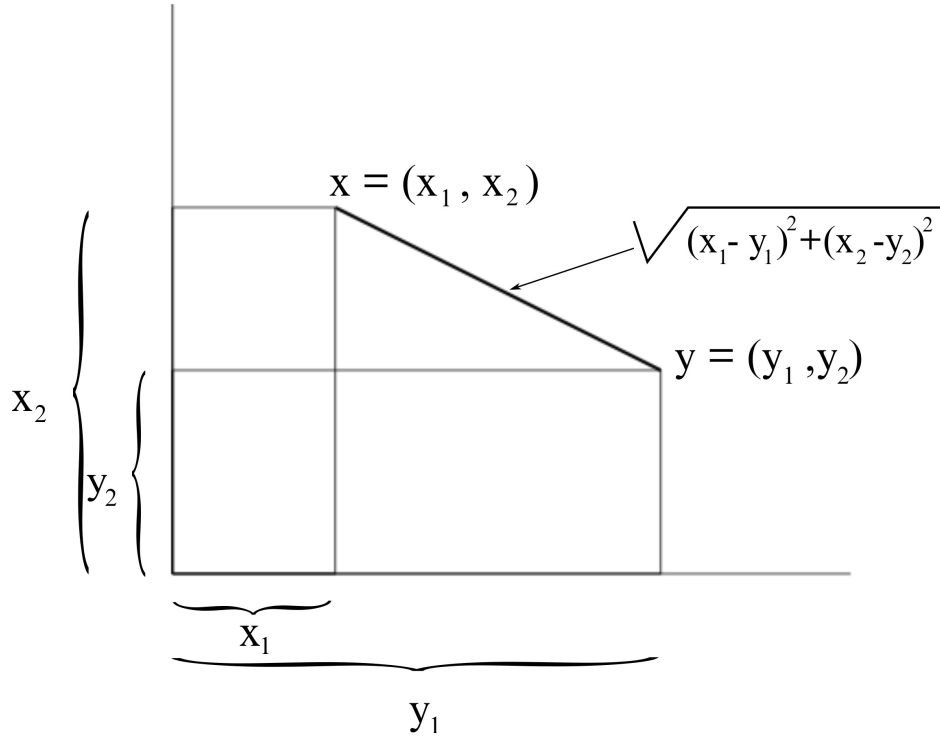


Figure 5.3: The Euclidean metric in \mathbb{R}^2 .

is 90 degrees (Fig. 5.3). Needless to say at this point that triangles, squares, or coordinates do not exist in the wild, in some kind of naturally created inorganic or organic form; even “inorganic” or “organic” are taxonomies of chemistry and biology, not of an observed nature. A triangle is an artefact produced by the artefact of a drawing stick, on the plane artefact of paper or papyrus: it is an assemblage, in other words, both material and phantasmatic. But this kind of triangle is in Euclidean plane and solid geometry. The triangle whose three vertices can be expressed in the form of coordinates on the Cartesian plane is an artefact of analytic geometry, not of the Euclidean one: there are, in other words, fundamental differences between Euclidean and analytic geometries. It is rather easy to prove that the Euclidean metric satisfies properties 1-3 of definition 1, page 276. The triangle inequality, the proof of which is not trivial, takes the following form:

$$\sqrt{\sum_{i=1}^n (x_i - y_i)^2} \leq \sqrt{\sum_{i=1}^n (x_i - z_i)^2} + \sqrt{\sum_{i=1}^n (z_i - y_i)^2} ,$$

where $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n), z = (z_1, \dots, z_n)$ are any points in \mathbb{R}^n ; in the case of $\mathbb{R}^1 = \mathbb{R}$, for any points x, y, z in \mathbb{R} , the inequality becomes:

$$|x - y| \leq |x - z| + |z - x|.^8$$

Let us now give an example of another interesting metric:

Example 2. *Considering again the set of n -tuples $x = (x_1, \dots, x_n)$ we can define the **Manhattan, or L^1 , distance**:*

$$\rho_{\text{Manhattan}}(x, y) = \sum_{i=1}^n |x_i - y_i| = |x_1 - y_1| + \dots + |x_n - y_n|,$$

where $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ any points of \mathbb{R}^n . The space \mathbb{R}^n together with the Manhattan (or L^1) distance is called the **n -dimensional L^1 space**,⁹ which is the simplest example of a **non Euclidean geometry**: the so-called **taxicab geometry**.

The triangle inequality now takes the following form:

$$\sum_{i=1}^n |x_i - y_i| \leq \sum_{i=1}^n |x_i - z_i| + \sum_{i=1}^n |z_i - y_i| ,$$

where $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n), z = (z_1, \dots, z_n)$ are any points in \mathbb{R}^n . In the case of \mathbb{R} the inequality becomes again:

$$|x - y| \leq |x - z| + |z - x|,$$

just like in the case of the Euclidean metric on \mathbb{R} . We will see the L^1 spaces later on. The geometry equipped with the Manhattan distance is called the **taxicab geometry**, because this is the problem of a taxi driver in Manhattan:¹⁰ the driving distance between two points in Manhattan is not the straight segment that connects these points, which is calculated by the Euclidean metric, but the sum of the absolute differences of their coordinates (Fig. 5.4);

⁸The $|x|$ is the *absolute value* of x , that is x without the minus or plus signs.

⁹In the Russian text, as well as in the English translation, this metric space is denoted by \mathbb{R}_1^n .

¹⁰Or in general any type of city plan in which streets run at right angles to each other.

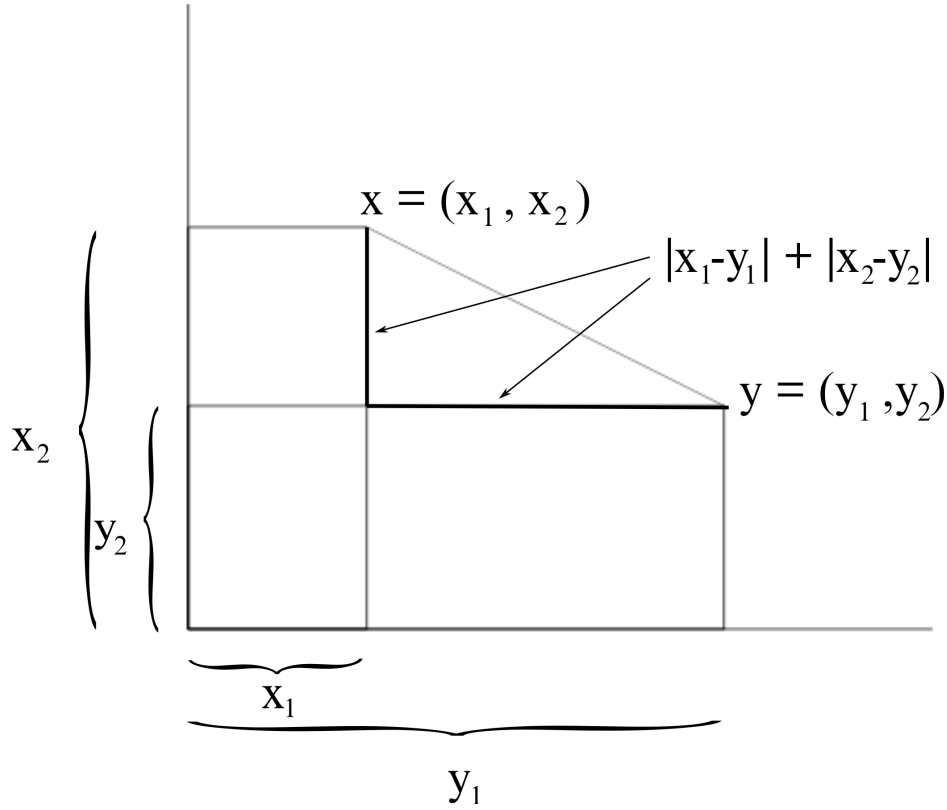


Figure 5.4: The Manhattan distance in \mathbb{R}^2 .

the well known from Euclidean geometry shapes, like the circle, change form, as well (see [255]).

We are going to see now an example in which the points of the metric space are functions.

Example 3. Let $[a, b]$ be a *closed interval*, that is the set whose numbers are between a and b ($a < b$), a and b included: $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$. The set $C_{[a,b]}$, or $C[a, b]$ in the Russian text, of all *continuous* functions $f : [a, b] \rightarrow \mathbb{R}$ ¹¹ becomes a metric space with the *uniform* metric:

$$\rho(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)| = \sup\{|f(x) - g(x)| \mid x \in [a, b]\}$$

where f, g continuous functions with the closed interval $[a, b]$ as their domain, and \mathbb{R} as their range: $f, g : [a, b] \rightarrow \mathbb{R}$, and \sup is their *supremum*, that is the

¹¹The graph of a continuous function has no “gaps”.

smallest number which is greater than or equal to each absolute difference of all their values in \mathbb{R} :

$$\sup_{x \in [a, b]} |f(x) - g(x)| \geq |f(y) - g(y)|, \text{ for all } y \in [a, b].$$

The supremum of a number set is always a number, and in the case of the supremum of the values of a continuous function defined on a closed interval it coincides with the maximum value of the function:

$$\sup_{x \in [a, b]} |f(x) - g(x)| = \max_{x \in [a, b]} |f(x) - g(x)|, \text{ since } f, g \text{ are continuous.}$$

A space like $C_{[a, b]}$, whose points are functions, is called a *function space*. The triangle inequality now becomes:

$$\sup_{x \in [a, b]} |f(x) - g(x)| \leq \sup_{x \in [a, b]} |f(x) - h(x)| + \sup_{x \in [a, b]} |h(x) - g(x)|,$$

where f, g, h lie in $C_{[a, b]}$; $C_{[a, b]}$ has become a *uniform* space. This is the starting point of *functional analysis*: the geometrical study of function spaces. We are not interested any more in the formula of a function, but in its properties.

What has been accomplished so far is to expand Euclidean geometry to non-Euclidean ones by Euclidean means: a metric space, due to the required properties of a metric still has triangles in the Euclidean sense: only the measurement of the distance between two points changes. And with the last example we can start realising something of major importance in functional analysis: the artefacts of geometrical figures now are not the “geometrical interpretation” of the theory any more, as is the case for examples 1 and 2; they can confuse rather, than illuminate the study of function spaces. The use of mathematical symbols now becomes indispensable, and the concepts more abstract. We can start now forming an idea of the mathematicians’ “spirit world”: we can use our (lay) imagination to approach their (shamanist) imagination.¹²

After the definition of a metric comes another very handy, and necessary indeed, definition:

¹²The mathematical symbols are now becoming the *magical runes* of these *shamans*.

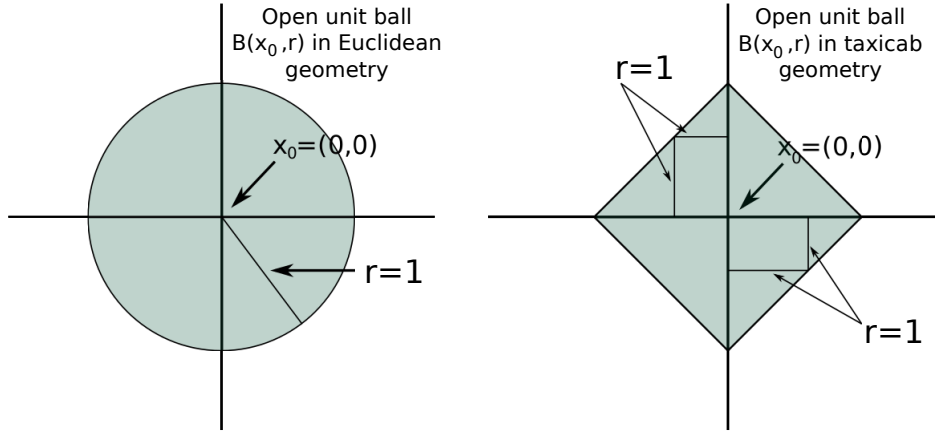


Figure 5.5: Euclidean and taxicab open balls in \mathbb{R}^2 .

Definition 2. Let (X, ρ) be a metric space and a point $x_0 \in X$; an *open ball* with *centre* x_0 and *radius* $r > 0$ is the set $B(x_0, r) = \{x \in X \mid \rho(x_0, x) < r\}$; a *closed ball* with *centre* x_0 and *radius* r is the set $B[x_0, r] = \{x \in X \mid \rho(x_0, x) \leq r\}$.

We already know the form of the open and closed balls in \mathbb{R} :

$$B(x_0, r) = (x_0 - r, x_0 + r) = \{x \in \mathbb{R} \mid |x - x_0| < r\}, \text{ which is an open interval};$$

$$B[x_0, r] = [x_0 - r, x_0 + r] = \{x \in \mathbb{R} \mid |x - x_0| \leq r\}, \text{ which is a closed interval}.$$

If the radius is one, then $B(x_0, 1)$ is the *open unit ball*, and $B[x_0, 1]$ is the *closed unit ball*, where $x_0 \in X$. In Fig.5.5 we can see the form of the open unit ball with centre the beginning $x_0 = (0, 0)$ of the coordinate axes in the 2-dimensional Euclidean and taxicab geometries. Open unit ball in two dimensions means the disc without the circumference, whereas the closed unit ball includes the circumference of the circle. In three dimensions the surface of the sphere does not belong to the open ball, while it is contained in the closed ball. In more than three dimensions we lose view of the materiality of the ball: we cannot perceive directly artefacts of four or more dimensions, if they exist.

All the physical realities which are more than four dimensions are always translated by scientists into artefacts of three or less, dimensions. In Fig.

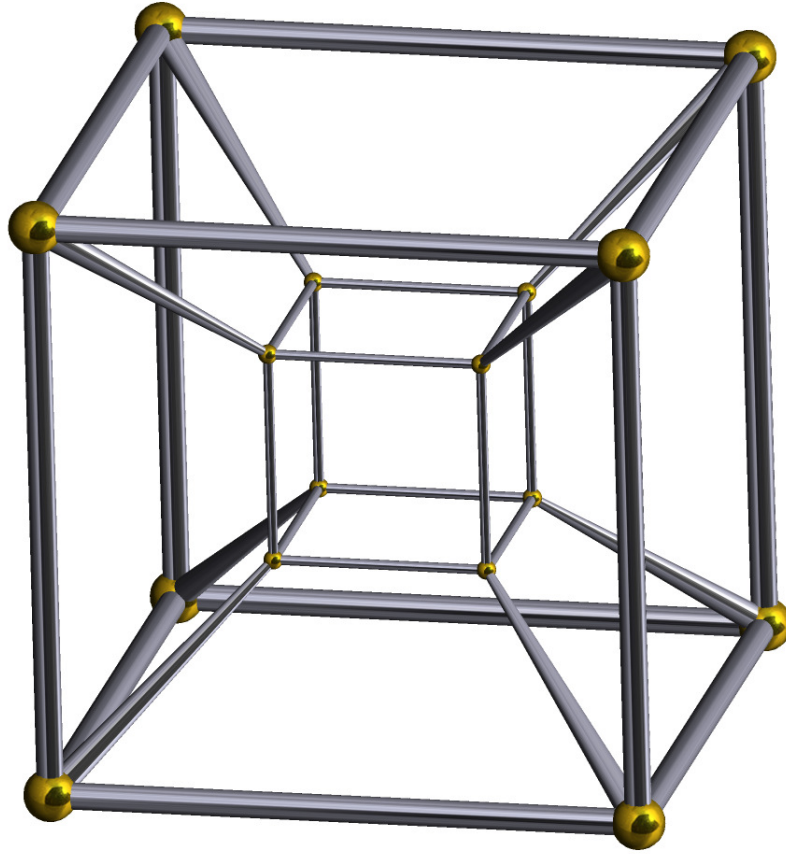


Figure 5.6: Schlegel diagram of a tesseract.

5.6, for example we can see the Schlegel diagram of a tesseract¹³, that is a 3-dimensional projection of a 4-dimensional cube; the artefact is actually 2-dimensional, but the human eye can perceive a third dimension in a drawing, which is the sense of depth, or perspective. The Schlegel diagram of a tesseract is, actually, a material attempt to extend the sense of perspective to more dimensions. In a recent movie called *Dredd*, a new drug appears in the black market of a future American megacity which slows down to an extreme degree the perception of time flow; the movie was filmed in 3D and the director utilizes the sense of spatial perspective of the viewer, to convey this slow-down of the perceived flow of time when under the influence of this new drug. In

¹³Taken from <http://en.wikipedia.org/wiki/Tesseract>.

the theory of relativity, both special and general relativity, time has become the fourth dimension. But still it cannot be perceived as a fourth dimension from first-hand experience; it can be experienced only through mathematical theory and its measurement is accomplished by instruments independent of human consciousness, which produce 2-dimensional artefacts.¹⁴ When it comes to a function space the situation is even worse: the (open or closed) balls can be (proved to be) infinite-dimensional, that is of countably infinite dimensions. The open unit ball in the uniform function space $C_{[a,b]}$ with centre the continuous function $f_0 : [a, b] \rightarrow \mathbb{R}$ now becomes:

$$B(f_0, 1) = \left\{ f \in C_{[a,b]} \left| \sup_{x \in [a,b]} |f_0(x) - f(x)| < 1 \right. \right\}.$$

These can indeed be difficult to handle, but still the mathematicians have managed to move ahead.

Definition 3. Let (X, ρ) be a metric space and a $\{x_n\}_{n \in \mathbb{N}}$ a sequence of points lying in X , that is $\{x_n\}_{n \in \mathbb{N}} \subseteq X$. Then $\{x_n\}_{n \in \mathbb{N}}$ *converges* to a point x_0 of X , denoted by $x_n \rightarrow x_0$,¹⁵ if, given any $\varepsilon > 0$, there is a natural number $n_\varepsilon \in \mathbb{N}$,¹⁶ such that the open ball $B(x_0, \varepsilon)$ contains all the points of the sequence beyond n_ε , that is $\{x_n\}_{n \geq n_\varepsilon} \subseteq B(x_0, \varepsilon)$. The point x_0 is the *limit* of the sequence $\{x_n\}_{n \in \mathbb{N}}$.

The introduction of limits to a generic metric space is the starting point of mathematical analysis. If $x_n \rightarrow x_0$, then $\lim_{n \rightarrow \infty} \rho(x_n, x_0) = 0$, and vice versa; that is the computation of the limit of a sequence in a generic metric space is *sub-delegated* to the computation of a usual limit in calculus. The ε is actually the error in computations, which is arbitrarily chosen by the mathematician, and therefore depends on a more general social context, since mathematics is not only a theoretical venture, but an applied discipline, as well (see [302]). Some examples will illustrate the concept of a limit.

¹⁴Actually only 3-dimensional artefacts can be perceived by the human eye; a sheet of white A4 paper seems to be 2-dimensional, because the third dimension of its thickness is almost invisible. A bacterium for the same reason appears to be dimensionless.

¹⁵While n tends to infinity, $n \rightarrow \infty$.

¹⁶The subscript ε denotes that n_ε is, in general, dependent on ε , while ε can be chosen arbitrarily.

Example 4. Let us consider the number $\sqrt{2}$, and a sequence $\{x_n\}_{n \in \mathbb{N}}$, such that $x_n \rightarrow \sqrt{2}$. Let $\varepsilon > 0$; then, according to the previous definition, there is a $n_\varepsilon \in \mathbb{N}$, such that $x_n \in B(\sqrt{2}, \varepsilon)$, for any $n \geq n_\varepsilon$. Then

$$|x_n - \sqrt{2}| < \varepsilon, \text{ for any } n \geq n_\varepsilon.$$

Suppose now we have the following first 10 terms of a sequence $\{x_n\}_{n \in \mathbb{N}}$, such that $x_n \rightarrow \sqrt{2}$:

$$\begin{array}{ll} x_1 = 1, & x_6 = 1,41421 \\ x_2 = 1.4, & x_7 = 1.414213, \\ x_3 = 1.41 & x_8 = 1.4142135, \\ x_4 = 1.414 & x_9 = 1.41421356, \\ x_5 = 1.4142, & x_{10} = 1.414213562; \end{array}$$

These are actually the first ten terms of the sequence A002193 in the On-Line Encyclopedia of Integer Sequences. If we want error $e < .001$ in the calculation of $\sqrt{2}$, we can set $\varepsilon = .001$, and therefore

$$|x_n - \sqrt{2}| < 0.001, \text{ that is } (x_n - 0.001)^2 < 2 \text{ \& } 2 < (x_n + 0.001)^2.$$

We check which of these last inequalities hold for each of the first ten terms of the sequence $\{x_n\}_{n \in \mathbb{N}}$:

$$\begin{array}{lll} n = 1, & (x_1 - .001)^2 = .9980010000, & (x_1 + .001)^2 = 1.002001000; \\ n = 2, & (x_2 - .001)^2 = 1.957201000, & (x_2 + .001)^2 = 1.962801000; \\ n = 3, & (x_3 - .001)^2 = 1.985281000, & (x_3 + .001)^2 = 1.990921000; \\ n = 4, & (x_4 - .001)^2 = 1.996569000, & (x_4 + .001)^2 = 2.002225000; \\ n = 5, & (x_5 - .001)^2 = 1.997134240, & (x_5 + .001)^2 = 2.002791040; \\ n = 6, & (x_6 - .001)^2 = 1.997162504, & (x_6 + .001)^2 = 2.002819344; \\ n = 7, & (x_7 - .001)^2 = 1.997098300, & (x_7 + .001)^2 = 2.002827835; \\ n = 8, & (x_8 - .001)^2 = 1.997172397, & (x_8 + .001)^2 = 2.002829251; \\ n = 9, & (x_9 - .001)^2 = 1.997172566, & (x_9 + .001)^2 = 2.002829420; \\ n = 10, & (x_{10} - .001)^2 = 1.997172572, & (x_{10} + .001)^2 = 2.002829426. \end{array}$$

We can clearly see that for $n \geq 4$ the inequality $|x_n - \sqrt{2}| < 0.001$ does hold. So we conclude that $n_{.001} = 4$. In other words by choosing any term x_n of the sequence, such that $n \geq 4$ we can have the desired approximation within the error limits we have set.

Decisions concerning the acceptable error in a measurement are not dictated by the needs of the mathematical community, but they are usually *extrasystemic imperatives*: for a risk analyst the approximation of two decimal points can be enough, since there are no smaller currency denominations; NASA may need an approximation of four decimals; a quantum physicist usually needs an approximation of 10 to 20 decimals. The theory of metric spaces is, in fact, an approximation theory in a generic space. It should be noted that in the approximation of $\sqrt{2}$ no reference was made as to whether the metric used for approximation was the Euclidean or the Manhattan one: in \mathbb{R} they both coincide with the absolute value of the difference of the values of the points at issue; that is

$$\rho_{\text{euclidean}}(x, y) = \rho_{\text{manhattan}}(x, y) = |x - y|,$$

where x, y any elements of \mathbb{R} .

The importance of the rational numbers, and the decimal ones in particular is illustrated by the following:

Example 5. *Every real m -tuple $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ ($m \in \mathbb{N}$) can be approximated by a rational sequence of the form $\{x^{(n)}\}_{n \in \mathbb{N}} = \{(x_1^{(n)}, \dots, x_m^{(n)})\}_{n \in \mathbb{N}} \subseteq \mathbb{Q}^m$ using either the Euclidean or the Manhattan metric (they are equivalent).*

We can see now an example of approximation in a function space:

Example 6. *The function $f(x) = 1 - \sqrt{1 - x}$, $x \in [0, 1]$, can be approximated by a sequence of polynomials $\{p_n\}_{n \in \mathbb{N}}$ on $[0, 1]$ with rational coefficients; where $p_n, n \in \mathbb{N}$, is a polynomial of m degree ($m \in \mathbb{N}$):*

$$p_n(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_2 x^2 + a_1 x + a_0, \quad x \in [0, 1].$$

The polynomials $p_n, n \in \mathbb{N}$ are defined on the closed unit interval $[0, 1]$, and we know from calculus that they are continuous and differentiable. They are

points, therefore, of the function space $C_{[a,b]}$. Using the uniform metric

$$\rho_u(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|,$$

where f, g any points of $C_{[a,b]}$, it will be shown that the following sequence of polynomials defined recursively is indeed such a sequence:

$$\begin{aligned} p_1(x) &= \frac{x}{2}, \quad x \in [0, 1] \\ p_{n+1}(x) &= \frac{1}{2}(p_n^2(x) + x), \quad x \in [0, 1] \quad (n \in \mathbb{N}). \end{aligned}$$

The proof will be conducted in a series of steps:

1. It can be shown, by using mathematical induction, that, when $x_0 \in [0, 1]$, we get $0 \leq p_n(x_0) \leq 1$, for all $n \in \mathbb{N}$: we consider x_0 as given, and then $\{p_n(x_0)\}_{n \in \mathbb{N}}$ becomes a number sequence which can be handled with a knowledge of calculus. The function $p_n(x), x \in [0, 1]$, therefore, is bounded¹⁷ for any $n \in \mathbb{N}$.
2. When $x_0 \in [0, 1]$, it can be shown by mathematical induction again that the sequence of the first order derivatives $\{p'_n(x_0)\}_{n \in \mathbb{N}}$ is a sequence of positive numbers: $p'_n(x_0) \geq 0$; therefore, considering now the function sequence $\{p_n\}_{n \in \mathbb{N}}$, we deduce that for every $n \in \mathbb{N}$, the function $p_n(x), x \in [0, 1]$, is monotonically increasing; we can now conclude that

$$\max_{x \in [0,1]} p_n(x) = p_n(1),$$

for any $n \in \mathbb{N}$.

3. Employing induction for a third time, by considering x_0 as a given point of $[0, 1]$, it can be shown that $p_{n+1}(x_0) \geq p_n(x_0), n \in \mathbb{N}$; the number sequence $\{p_n(x_0)\}_{n \in \mathbb{N}}$, in other words, is monotonically increasing, for each $x_0 \in [0, 1]$.
4. Since by steps 1 and 3, given an $x_0 \in [0, 1]$, the number sequence $\{p_n(x_0)\}_{n \in \mathbb{N}}$ is monotonically increasing and bounded, we deduce that

¹⁷A function f is *bounded* when there is a number $c > 0$, such that $|f(x)| \leq c$, for all x ; then f is *bounded by c* , and c is a *bound* of f .

it converges to a point $\ell_0 \leq 1$. Using the recursive definition of the sequence of polynomials $\{p_n\}_{n \in \mathbb{N}}$, we deduce that $\ell_0 = 1 \pm \sqrt{1 - x_0}$. Since $\ell_0 \leq 1$, we conclude that $\ell_0 = 1 - \sqrt{1 - x_0} = f(x_0)$, that is

$$\lim_{n \rightarrow \infty} p_n(x_0) = f(x_0).$$

In other words the function sequence $\{p_n\}_{n \in \mathbb{N}}$ converges *pointwise* to f , that is point by point, and not uniformly, which is the desired type of convergence.

5. We will now zoom in on the fifth and most important step of the proof. Let $\varepsilon > 0$ be any positive number.

- (a) If $x_0 \in [0, 1]$, then we saw that $\lim_{n \rightarrow \infty} p_n(x_0) = f(x_0)$, or, in the usual notation of functional analysis for sequence limits, $p_n(x_0) \rightarrow f(x_0)$.
- (b) Taking into consideration the definition of convergence of a sequence in a metric space (Def. 3, page 284), for $\frac{\varepsilon}{2} > 0$ there is a $n_{\frac{\varepsilon}{2}} \in \mathbb{N}$, such that $p_n(x_0) \in B(f(x_0), \frac{\varepsilon}{2})$, when $n \geq n_{\frac{\varepsilon}{2}}$.
- (c) But the open ball $B(f(x_0), \frac{\varepsilon}{2})$ is a ball on the number space \mathbb{R} , which is not the space we want, that is the function space $C_{[0,1]}$.
- (d) Since $p_n(x_0) \in B(f(x_0), \frac{\varepsilon}{2})$, for $n \geq n_{\frac{\varepsilon}{2}}$, it means that

$$|p_n(x_0) - f(x_0)| < \frac{\varepsilon}{2}, \text{ when } n \geq n_{\frac{\varepsilon}{2}}.$$

- (e) But x_0 is any point in $[0, 1]$, and therefore the positive number $\frac{\varepsilon}{2}$ is an upper bound for the function $|p_n x - f(x)|, x \in [0, 1]$, that is,

$$|p_n(x) - f(x)| \leq \frac{\varepsilon}{2}, \text{ for all } x \in [0, 1], \text{ when } n \geq n_{\frac{\varepsilon}{2}}$$

- (f) Therefore the last inequality will hold for the *least upper bound* of the function, as well, that is its *supremum* :

$$\sup_{x \in [0,1]} |p_n(x) - f(x)| \leq \frac{\varepsilon}{2}, \text{ when } n \geq n_{\frac{\varepsilon}{2}}$$

- (g) Since the function $|p_n(x) - f(x)|$, $x \in [0, 1]$, is continuous and defined on the closed interval $[0, 1]$, for all $n \in \mathbb{N}$, it has a maximum which coincides with its supremum:

$$\max_{x \in [0, 1]} |p_n(x) - f(x)| \leq \frac{\varepsilon}{2}, \text{ for all } n \geq n_{\frac{\varepsilon}{2}}$$

6. We have reached therefore the conclusion that

$$\rho_u(p_n, f) < \varepsilon, \text{ for all } n \geq n_{\frac{\varepsilon}{2}},$$

where ρ_u is the uniform metric of the uniform function space $C_{[0, 1]}$;

7. In other words, that for any $\varepsilon > 0$, there is an $n_\varepsilon (= n_{\frac{\varepsilon}{2}}) \geq n$, such that $p_n \in B(f, \varepsilon)$, for all $n \geq n_\varepsilon$, where $B(f, \varepsilon)$ is the open ball in $C_{[0, 1]}$,¹⁸ with centre the function f and radius $\varepsilon > 0$.
8. Considering definition 3, page 284, from the above we deduce that the sequence of polynomials $\{p_n\}_{n \in \mathbb{N}}$ *converges uniformly to f* .

Every proof has a particular audience: a theorem is not proved in the same way when it is intended for undergraduates, as when it is intended for peers. Undergraduates, and later postgraduates are not peers; postgraduates in particular are nor peers *yet*. The first thing to mention is that the first three steps can be solved by knowledge of calculus. It has to be noted that there is a play with the variables: the x variable is stabilised and the n runs through \mathbb{N} , steps 1, 2, and 3, and then the n variable is stabilised and the x runs through $[0, 1]$, steps 1 and 2. It is important, therefore, to do, as a reader, this interchange on the variables mentally, and to be in a position to reproduce it later in examinations by hand. For a sociologist of science, it is more important not to understand a proof, than to understand it: it simply poses the question of how the subjects of a proof understand, and, moreover, how these subjects communicate with others their results. The usual answer that pervades to that is that someone is good at mathematics because he or she is especially talented: the really talented people are actually very few,

¹⁸Which means $B(f, \varepsilon) \subseteq C_{[0, 1]}$.

and generally they receive prizes for their innovations.¹⁹ The above proof can be very daunting to an undergraduate, but all of the postgraduates who specialise in a branch of mathematical analysis, or probability theory, are able to understand it. So how can someone understand this proof?

An important way, in fact, of understanding a mathematical proof is something common sense dictates: reading the above proof. But before proceeding to read the proof, it has to be reminded that a proof in the beginning was not a proof: it was a passing thought, an idea, then it was an experiment, a discussion with colleagues, and then it was tidied up for presentation to a peer journal. So when we read a proof in a journal, we see the presentable version of it, not its process of generation. In reading the above proof it will be assumed that the imaginary reader of it is proficient in the English language, although this is not always the case. The first thing to bear in mind is *working memory*: the “system of interacting components [of the brain] that maintain newly acquired and reactivated stored information, both verbal and nonverbal, and make it available for further information processing”[38]. That is, in other words, the memory mechanisms *being used* to perform a task *at hand*. In steps 1, 2 and 3 the method used for proving is mathematical induction, which is already known from first year calculus. Depending on how many exercises a first year student has solved,²⁰ which involve induction, these sub-proofs will induce corresponding feelings of familiarity: some will understand it on the spot, just by reading it; some will have to copy the proof to a sheet of paper and try to do the proof themselves by hand; some will ask a fellow student, or the lecturer. In every case the result will be better understanding along with an increased sense of familiarity. The really difficult task for an early undergraduate will be to understand the transition from the number space \mathbb{R} , a space whose elements are already familiar from school, to the abstract function space $C_{[0,1]}$, whose elements are functions. The same holds for the transition from the number ball $B\left(f(x_0, \frac{\varepsilon}{2})\right)$ to the function

¹⁹The Fields medal is the most prestigious prize in mathematics, awarded to two up to four mathematicians who are not over forty years of age at the time of the award ceremony, during each International Congress of the International Mathematical Union, organised every four years.

²⁰Depending also on the student’s mathematical talent.

ball $B(f, \varepsilon)$. These happen in step 5. During this step the working memory may hinder perception, since it has to be modified, in order to identify new kinds of spaces and balls, that are necessary to proceed in functional analysis. This extra effort required to understand an extended concept of space, by modifying information processed in the working memory brain modules, makes an exercise a personal achievement, maybe small, maybe important; many similar exercises make this achievement business as usual, a trivial, everyday matter.

In general, returning to the previous proof, “*every function $f \in C_{[a,b]}$, a and b lying in \mathbb{R} , can be uniformly approximated by a sequence of polynomials with rational coefficients*”: this is the so-called *Stone-Weierstrass theorem*. Moreover there is something else about the proof that has not been mentioned: step 2, actually is not necessary, since its argument is covered by steps 5e, 5f, and 5g: that the function $p_n(x), x \in [0, 1]$, has a maximum for all $n \in \mathbb{N}$, or all $n \geq n_{\frac{\varepsilon}{2}}$. This proof was performed by the author by copying initially a proof available on the internet, and then translating into a more understandable for him framework. The argument of step 2 was included in the initial online proof. Later it became clear, that the modified proof of the author was valid, as well. This step was kept for two reasons. The first is to show that the initial reading of a proof is experimental, as to the understanding of it, especially when introducing new concepts. The second, and more important reason, is to demonstrate the importance of instruction, in the case of undergraduates and postgraduates, and peer reviewing, in the case of working mathematicians. The undergraduates are being socialized in the formal presentation of proofs by instruction and submission of assignments to be corrected, as well, as they usually miss or add unnecessary steps in a proof. In the case of working mathematicians, due to the nature of innovation in a much more complex framework, especially in articles crossing the various fields of mathematics, the possibility to miss a step in a proof is much higher. The above cited proof, obviously, can still have mistakes or omissions, since the author is not a professional mathematician.

Returning now to Kolmogorov and Fomin’s book, we have to cite some more important definitions, before we proceed to the next section.

Definition 4. Let (X, ρ) be a metric space and Y be a subset of X , i.e. $Y \subseteq X$. Then Y is **dense in** X , if every point x_0 of X , i.e. $x_0 \in X$, is the limit of a sequence $\{y_n\}_{n \in \mathbb{N}}$ in Y , i.e. $\{y_n\}_{n \in \mathbb{N}} \subseteq Y$, that is $y_n \rightarrow x_0$. If, moreover, Y is countable, that is $\text{card}(Y) = \aleph_0$, then X is **separable**.

We can say now that the set of rational numbers \mathbb{Q} is dense in \mathbb{R} , and, since \mathbb{Q} is countable, that \mathbb{R} is separable. Alternatively, \mathbb{Q} is a *dense sequence* in \mathbb{R} . Also the Stone-Weierstrass theorem actually asserts the set of polynomials of one variable with rational coefficients, denoted by $\mathbb{Q}[X]$ in algebraic geometry, is dense in $C_{[a,b]}$. Since $\mathbb{Q}[X]$ is countable, $C_{[a,b]}$ is separable. The importance of separable spaces cannot be overestimated: the majority of literature in functional analysis, especially its applied version, is on separable spaces. We do not know, and will never know, the exact value of $\sqrt{2}$, π , or Euler's number e ; we know only their rational approximations. The same holds for functions: if we do not know the exact formula of a function continuous on a closed interval, we can set the approximation error, and then find a corresponding polynomial to use as the function's formula. In other words, in the definition of a separable space in functional analysis a systemic imperative is latent: to study spaces whose points can be computationally approximated by sequences. This was the imperative underlying the early development of functional analysis. Later on, especially after the second world war, research turned also to more general spaces, and the concept of convergence was achieved by using only set theory, and doing away with the metrics and introducing instead topologies:

Definition 5. If X is a metric space with metric ρ , then the sets which are (countable or uncountable) unions of open balls $B(x, \varepsilon)$, $x \in X, \varepsilon > 0$, are called **open sets**, and the collection of open sets is the **topology** of X with respect to the metric ρ , denoted by $\mathfrak{T}_\rho(X)$, or \mathfrak{T}_ρ if X is known, or simply \mathfrak{T} . A closed set A , $A \subseteq X$, is a set whose **complement** $A^C = X \setminus A$ is **open**.

Unions of open balls are open sets. Closed balls are closed sets, as well as any intersection of them. Any union (countable or uncountable) of open sets is always an open set, and any finite intersection of open sets is always an open set. Any intersection (countable or uncountable) of closed sets is a closed set,

as well as any finite union of closed sets. A collection \mathfrak{T} of subsets of X is a topology, if: (a) \emptyset and X lie in \mathfrak{T} ; (b) the union of the elements of any family of elements of \mathfrak{T} is an element of \mathfrak{T} , that is, $\{A_i\}_{i \in I} \subseteq \mathfrak{T}$ implies $\bigcup_{i \in I} A_i \in \mathfrak{T}$, where the index set I can be countable or uncountable; (c) $A \in \mathfrak{T}$ and $B \in \mathfrak{T}$ implies $A \cap B \in \mathfrak{T}$. Then (X, \mathfrak{T}) is a *topological space*. A topology defined on a set X does not necessarily correspond to a metric. Kolmogorov and Fomin's book have a whole chapter on general topological spaces, which is a standard subject for postgraduate students in mathematical analysis, but this theme will not be further pursued here. It is necessary, nevertheless, for the study of probability and stochastic processes on general topological spaces.

After we have seen some examples of convergence on metric spaces we now need a criterion as to whether a sequence converges or not:

Definition 6. A sequence of points $\{x_n\}_{n \in \mathbb{N}}$ in a metric space (X, ρ) is called a *Cauchy (or fundamental) sequence* if it satisfies the *Cauchy criterion*: given any $\varepsilon > 0$, there is a number $n_\varepsilon \in \mathbb{N}$ such that $\rho(x_n, x_m) < \varepsilon$, for all $n, m > n_\varepsilon$. A metric space (X, ρ) is called *complete*, if every Cauchy sequence is convergent.

If a $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence, then it is always convergent (see Fig. 5.7). The opposite does not always hold. We saw for example, that there is a rational sequence which converges to $\sqrt{2}$, but $\sqrt{2}$ is not rational, and so \mathbb{Q} is not complete, whereas \mathbb{R} is. The importance of completeness lies in the fact that in a complete space there are no structural holes. The set of rational numbers, for example, has holes: $\sqrt{2}$ is such a hole. But any incomplete space can always be extended to a complete one with respect to its metric, like \mathbb{Q} , for instance, can be extended to \mathbb{R} ; When a new space (X', ρ') is a complete extension of an incomplete one (X, ρ) , it is called the *completion* of (X, ρ) .

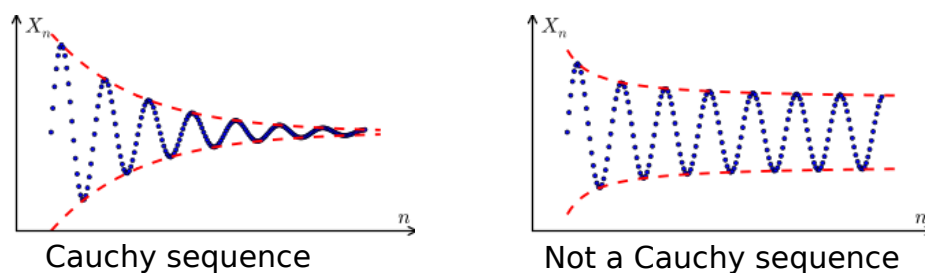


Figure 5.7: A Cauchy and a non-Cauchy sequence.

5.5 Straight Line Management

In the previous section the metric (or distance) function was presented and it was shown how it is used to define limits. In fact, the concept of a limit is what defines mathematical analysis, or analytical techniques, as they are often termed, as well. Without convergence, actually, there would be almost no mathematics today. The proofs that were given were rather too detailed at some points; this is what is called *mathematical rigour*: extreme attention to detail, to the point of appearing to a lay person sometimes as a symptom of obsessive-compulsive behavioral disorder. But it serves a fundamental purpose: since there is no way in a generic function space to consult a geometrical model on how to investigate that space, the mathematical analyst has to be extremely careful on handling formulas, concepts and proofs. Although the situation is similar to a toddler building with Lego bricks, the Lego bricks, in the case of functional analysis, are *phantasms*,²¹ seen only by well trained shamans. It is not accidental that, while undergraduate calculus books have many graphic figures, advanced books on functional analysis have very few, if any; the reader is assumed to “understand” the object of study. Due to many computations by hand, in exercises, proofs, and so on, the undergraduate gradually becomes able to *see* the meaning, to *watch* the unfolding narrative of a proof, especially when geometrical properties are involved. This is the same situation when toddlers learn a language very early in their lives; in that case the toddler learns to *hear* the meaning, and to

²¹That is illusions, apparitions, or ghosts, according to the Oxford dictionary definition of the noun “phantasm”.

listen to the unfolding narrative of a story or a fairy tale. But seeing and hearing are actually *material modalities*, that is, ways of engaging with the environing material space by means of *reading*. Building and talking are, on the other hand, material modalities by means of *writing*, which are based, correspondingly, on watching and listening:

Humans ... can make tools as well as [sound] symbols, both of which derive from the same process, or, rather, draw upon the same basic equipment in the brain. This leads us back to conclude, not only that language is as characteristic of humans as are tools, but also that both are the expression of the same intrinsically human property ([281, p.113]; see also [421, p.164-169]).

Proceeding now to chapter 4 of Kolmogorov and Fomin's book, we are introduced to *linear* (or *vector*) spaces: the points of a set X , which are now called *vectors*, can be added to and subtracted from one another, and they can be multiplied by usual numbers, which are also called *scalars*. We say then that X is a *real linear space*.²² The set $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}, i = 1, \dots, n\}$ ($n \in \mathbb{N}$), becomes a linear space if we define the addition of vectors, and the product of scalar with a vector as follows:

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

$$\alpha(x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n),$$

where $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)$ vectors of \mathbb{R}^n , and $\alpha \in \mathbb{R}$ (see Fig. 5.8).

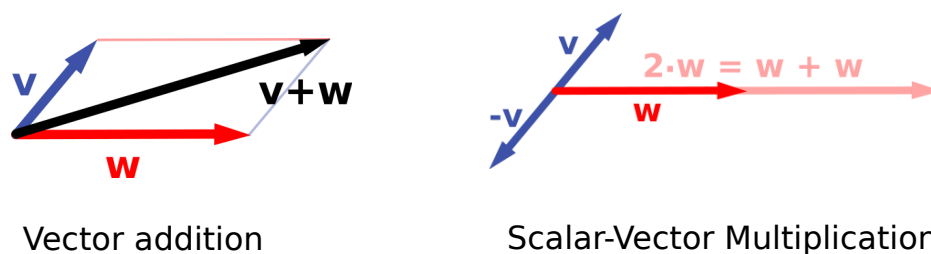


Figure 5.8: Vector addition, and scalar-vector product in \mathbb{R}^2 .

²²Or a *complex linear space* if the scalars are complex numbers.

The space $C_{[a,b]}$ of all continuous functions $f : [a, b] \rightarrow \mathbb{R}$, where a, b are real numbers, equipped with the usual addition between functions, and the usual multiplication between a number and a function, becomes a linear space as well. It must be noted that the multiplication of a scalar with a vector produces a vector, and not a scalar. So the multiplication of the number α with the function $f \in C_{[a,b]}$ produces the new function $\alpha f \in C_{[a,b]}$ defined as $(\alpha f)(x) = \alpha f(x)$, $x \in \mathbb{R}$. A vector x in undergraduate calculus is usually denoted by \vec{x} to mark its different treatment from a number, but in functional analysis this notation is dropped. So the *null* or *zero* vector, which in calculus is denoted as $\vec{0}$, in functional analysis it is denoted by the same symbol used for the number zero, that is “0”. If there are many null vectors from different spaces, then the notation 0_X can be used to denote the null vector of the linear space X , in order to avoid any confusion. So the null vector of $C_{[a,b]}$ is

$$0_{C_{[a,b]}}(x) = 0_{\mathbb{R}}, x \in [a, b].^{23}$$

Given a linear space X , a set Y is called a *linear subspace* of X , if $Y \subseteq X$, and for any real numbers α and β and any vectors x and y lying in Y , the *linear combination* $\alpha x + \beta y$ lies in Y , that is $\alpha x + \beta y \in Y$ (see Fig. 5.9). A linear space is always a subspace of itself. A straight line L in a generic vector X space determined by two points x, y in X is defined as $L = \{\alpha x + \beta y \mid \alpha \in \mathbb{R}, \beta \in \mathbb{R}\}$. A linear space, therefore, contains the straight lines determined by all the pairs of its points. The space $P_{[a,b]}$ of polynomials defined on the closed interval $[a, b]$ is a linear subspace of $C_{[a,b]}$. If $B \subseteq X$, the *linear hull* of B , denoted by $[B]$, is the set of all (finite) linear combinations of vectors of B :

$$[B] = \left\{ \sum_{i=1}^n \alpha_i x_i \mid \alpha_i \in \mathbb{R}, x_i \in B, i = 1, 2, \dots, n, n \in \mathbb{N} \right\}$$

$[B]$ is always a linear subspace of X ; if $[B] = X$, then B *generates* X ; if B generates X and every vector in B , cannot be written as a linear combination of other vectors in B , then it is called a *basis* of X . The cardinal number of

²³We can see now that symbols have already become arcane to the lay person, that is, they have become the academic runes of the shamans of mathematics.

a basis of a linear space X is called the *dimension* of X . There are infinitely many bases of a linear space but they all have the same cardinal number, either the bases are finite or infinite.

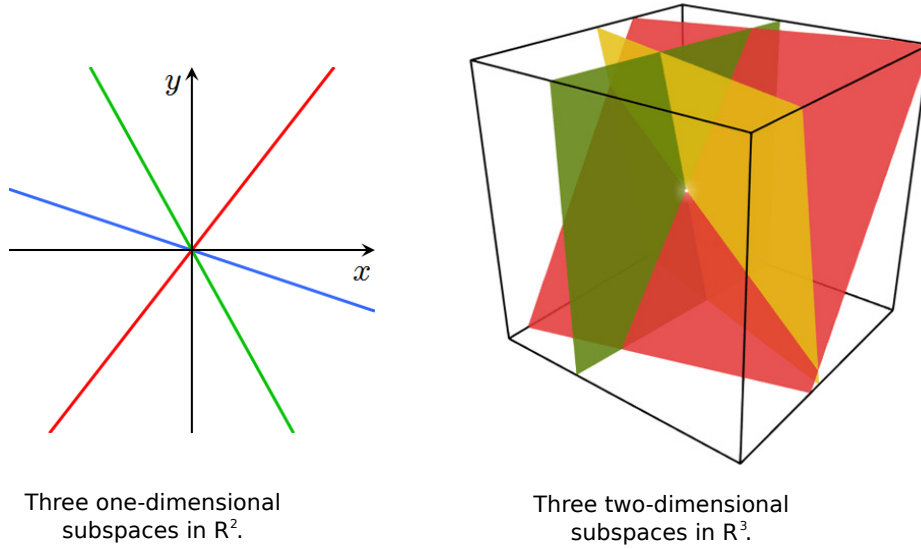


Figure 5.9: One- and two-dimensional subspaces.

A linear space X can become a metric space by defining a *norm function*:

Definition of a norm function

Definition 7. Let X be a linear space. A *norm function* on X , $\|\cdot\|: X \rightarrow$ is defined as follows:

$$\|x\| \geq 0 \quad (\text{non-negativity}) \quad (5.5)$$

$$\|x\| = 0 \Leftrightarrow x = 0_X \quad (\text{nondegeneracy}) \quad (5.6)$$

$$\|\alpha x\| = |\alpha| \|x\| \quad (\text{positive homogeneity}) \quad (5.7)$$

$$\|x + y\| \leq \|x\| + \|y\| \quad (\text{triangle inequality}) \quad (5.8)$$

where x, y are any vectors in X , and α any number in \mathbb{R} .

We can see that properties 5.5, 5.6 and 5.8 correspond to the properties 5.1, 5.2 and 5.4 of a metric (definition 1, page 276). Since a norm of a vector is considered its “length”, we would expect positive homogeneity (property 5.7):

the length of a vector multiplied by a number is equal to the product of the number without the minus or plus signs²⁴ times the length of the original vector as issue. The metric now can be defined as $\rho(x, y) = \|x - y\|$, where x, y any vectors in X . The linear space X equipped with a norm is called a *normed linear space*, and it is formally denoted by $(X, \|\cdot\|)$. A normed linear space is automatically a metric space; it is actually a special case of a metric space. A complete normed linear space is called a *Banach space*.²⁵ Everything said about metric spaces so far holds for normed linear spaces, as well.

Now we are getting close to redefine Euclidean geometry on infinite-dimensional terms; there are, though still a few more mathematical entities left to be defined.

Definition 8. Let X be a real linear space. A *scalar (or inner) product* on X is a function $\langle \cdot, \cdot \rangle: X \times X \rightarrow \mathbb{R}$ with the following properties:

$$\begin{cases} \langle x, x \rangle \geq 0 \\ \langle x, x \rangle = 0 \Leftrightarrow x = 0_X \end{cases} \quad (\text{positive definiteness}) \quad (5.9)$$

$$\langle x, y \rangle = \langle y, x \rangle \quad (\text{symmetry}) \quad (5.10)$$

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \quad (\text{linearity}) \quad (5.11)$$

where x, y, z are any vectors in X and α, β are any scalars in \mathbb{R} . A linear space X equipped with an inner product is called a *Euclidean space*.

A Euclidean space X becomes a normed linear space by setting $\|x\| = \sqrt{\langle x, x \rangle}$, for all $x \in X$.

If the property of positive definiteness is relaxed, and only the *non-degeneracy* of the inner product is kept, that is

$$\langle x, x \rangle = 0 \Leftrightarrow x = 0_X,$$

for all $x \in X$, then that marks the beginning of *semi-Riemannian geometry*,²⁶ a special case of which is the four dimensional Minkowski spacetime of

²⁴That is, the absolute value of the number $|\cdot|$.

²⁵Named after the Polish mathematician Stefan Banach

²⁶Or *pseudo-Riemannian geometry*.

Einstein's special relativity theory. Then there are three kind of vectors, or rather *events*, if we borrow the terminology of relativity:

Definition 9. *The causal character of an event x lying in a spacetime X depends on whether x is*

$$\begin{aligned} & \textit{spacelike}, \quad \text{if } \langle x, x \rangle > 0 \quad \text{or} \quad x = 0, \\ & \textit{lightlike}, \quad \text{if } \langle x, x \rangle = 0 \quad \text{and} \quad x \neq 0, \quad \text{or} \\ & \textit{timelike}, \quad \text{if } \langle x, x \rangle < 0. \end{aligned}$$

Lightlike events are also called *null*, and the set of lightlike events is the *light cone*, or the *null cone* (see Fig. 5.10). The causal character of an event means that events within the light cone cannot influence events outside the light cone, and vice versa; in other words, a particle can move either within, or outside the light cone; it cannot cross it. The norm of a vector in the case of semi-Riemannian geometry, is defined as

$$\|x\| = \sqrt{|\langle x, x \rangle|}, \quad x \in X.$$

Returning back to Euclidean spaces, because of the *Cauchy-Schwarz-Bunyakovski inequality* (C-S-B),²⁷ which asserts that if x, y are any vectors in the inner product space X , then the inequality $|\langle x, y \rangle| \leq \|x\| \|y\|$ is always valid, we can define the angle ϕ of x and y with cosine

$$\cos \phi = \frac{\langle x, y \rangle}{\|x\| \|y\|}, \quad x \neq 0, \quad y \neq 0,$$

since by C-S-B inequality

$$-1 \leq \frac{\langle x, y \rangle}{\|x\| \|y\|} \leq 1, \quad x \neq 0, \quad y \neq 0.$$

If $\|x\| = 1$, then x is called *normal*; if $\langle x, y \rangle = 0$, then x and y are *orthogonal*, that is, perpendicular to each other. A set of nonzero vectors $\{x_i \in X \mid i = 1, 2, \dots, n\}$ is called an *orthonormal system* if

$$\langle x_i, x_j \rangle = \begin{cases} 0, & \text{when } i \neq j \\ 1, & \text{when } i = j. \end{cases}$$

²⁷In the English text it is only Schwarz; in the Russian it is Cauchy-Bunykovski.

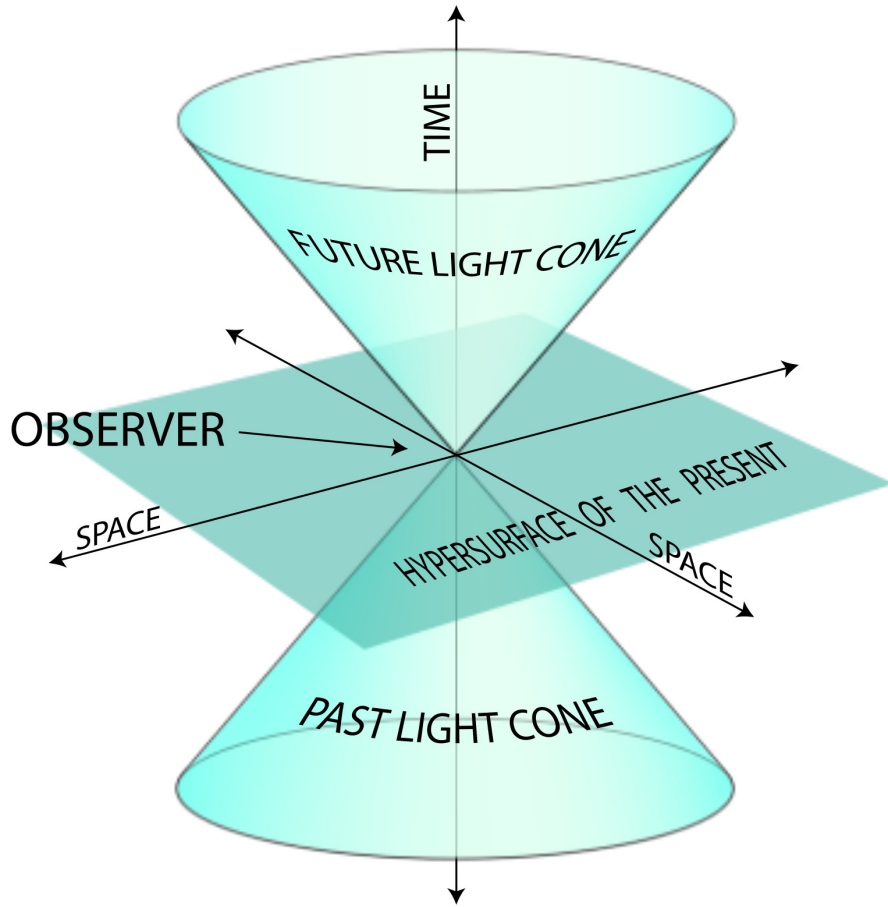
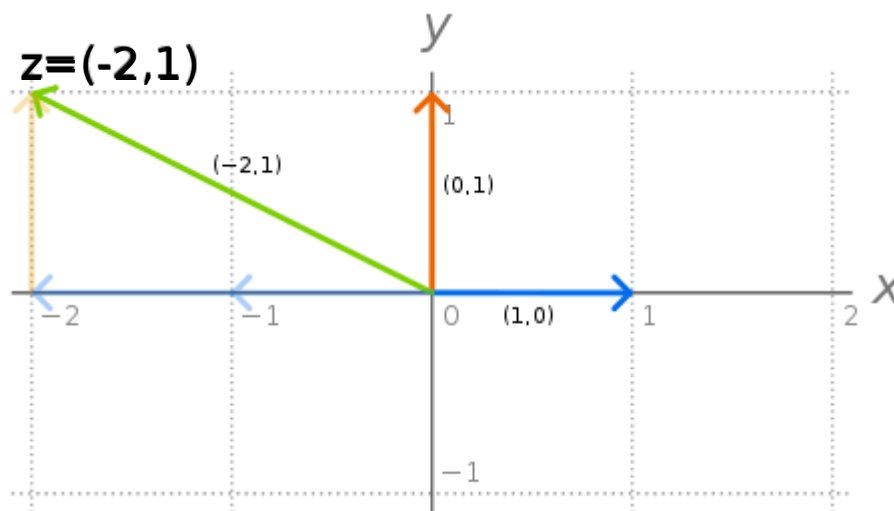


Figure 5.10: The light cone in the (non Euclidean) \mathbb{R}^3 spacetime.

The main interest in Euclidean spaces lies in *orthonormal bases* (see Fig. 5.11). Every basis, either finite or countable, can always be changed into a new orthonormal one, and every orthonormal system can be extended to an orthonormal basis.

A very interesting example mentioned in Kolmogorov-Fomin's book is the following:

Example 7. The space $C^2_{[a,b]}$ consisting of all the continuous functions defined on the closed interval a, b , a, b real numbers becomes a Euclidean when



$$z = (-2)(1,0) + (+1)(0,1).$$

Figure 5.11: The orthonormal base $\{(1, 0), (0, 1)\}$ in \mathbb{R}^2 .

equipped with the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx, \quad f, g \in C_{[a,b]}^2.$$

The norm of this space is

$$\|f\| = \sqrt{\langle f, f \rangle} = \left(\int_a^b f^2(x)dx \right)^{\frac{1}{2}},$$

and a very important orthogonal basis is the system of trigonometric functions

$$\begin{cases} e(x) = 1, & x \in [a, b] \\ f_n(x) = \cos \frac{2\pi nx}{b-a}, & x \in [a, b] \quad (n \in \mathbb{N}) \\ g_n(x) = \sin \frac{2\pi nx}{b-a}, & x \in [a, b] \quad (n \in \mathbb{N}) \end{cases}$$

since:

$$\begin{aligned}
\langle e, f_n \rangle &= \int_a^b e(x) \cdot f_n(x) dx = \int_a^b \cos \frac{2\pi nx}{b-a} dx = 0 \quad (n \in \mathbb{N}), \\
\langle e, g_n \rangle &= \int_a^b e(x) \cdot g_n(x) dx = \int_a^b \sin \frac{2\pi nx}{b-a} dx = 0 \quad (n \in \mathbb{N}), \text{ and} \\
\langle f_n, g_n \rangle &= \int_a^b f_n(x) \cdot g_n(x) dx = \int_a^b \cos \frac{2\pi nx}{b-a} \sin \frac{2\pi nx}{b-a} dx = 0 \quad (n \in \mathbb{N}).
\end{aligned}$$

In other words, for all $n \in \mathbb{N}$, the vectors e, f_n, g_n , are pairwise orthogonal, that is, pairwise perpendicular to each other.

Now we are ready to enter the universe of infinite-dimensional Euclidean geometry:

Definition 10. A *Hilbert space*²⁸ is a complete, separable, and infinite-dimensional Euclidean space.

If H is a Hilbert space and M is a subspace of H , then the *orthogonal complement* of M , denoted by M^\perp , is the subset of H whose vectors are orthogonal to all the vectors of M :

$$M^\perp = \{f \in H \mid \langle f, x \rangle = 0, \text{ for every } x \in M\}.$$

The orthogonal complement M^\perp of a subspace M is itself a Euclidean, or Hilbert, space and H is the *direct sum* of M and M^\perp , denoted by $H = M \oplus M^\perp$, that is, every vector $f \in H$ can be written in a unique way as

$$f = x_1 + x_2, \text{ for some } x_1 \in M, \text{ and } x_2 \in M^\perp.$$

Hilbert spaces have had a tremendous impact on twentieth century mathematics: complete means that any Cauchy sequence converges; separable means that there is a sequence such that every vector in the space is a limit of one of its subsequences; infinite-dimensional means that its bases are infinite. Since a Hilbert space is separable, there is a theorem that says that its

²⁸Named after the German mathematician David Hilbert.

bases will always be countable. Each basis, therefore, is a sequence as well. And the most natural question that now arises, is how is it possible for mathematicians to prove theorems that cannot have any material and artefactual representation whatsoever? How is it possible to prove theorems holding in infinite-dimensional spaces using only three-dimensional artefacts, or rather two-dimensional sheets of paper with written symbols on them, then using these theorems to formulate theories describing the quantum reality, since in quantum physics there are infinite degrees of freedom, that is, infinite dimensions, and then conduct experiments based on these theories with particle accelerators? It is impossible to explain all these without assuming the ability of *Homo faber*, not only to construct, but also to imagine.

The only modern Western discipline that has studied extensively imagination is psychoanalysis with its concept of phantasy. In modern Islamic philosophy, on the contrary, imagination (and creativity) is a recurring theme,²⁹ because any material representation of Allah is a sin; the faithful have to imagine God, to raise their consciousness beyond the artefact, in order to avoid the trap of idolatry. If someone goes into a Roman Catholic or into an Eastern Orthodox church, they will see depictions of saints, angels, and God: the Catholic paintings are more vivid than the Orthodox ones, but they both attempt to materially depict God. In a mosque, on the contrary, there is no visual depiction of Allah, only Arabic (usually) calligraphic script praising Him. The situation in mathematics of infinite-dimensional spaces is not much different: the mathematician has to go beyond the artefact in order to prove a theorem, not because it is a sin to do so, but because there are no infinite-dimensional artefacts to use, and if they do exist, the human eye has its limitations on that. Imagination, in other words, as the universe of human consciousness, undertakes the job of pushing perception beyond the artefact.

²⁹See for example [443, p.173-203].

5.6 The Pure and the Applied Sciences

Integral equations, an important field in both physics and mathematics, was the second major inspiration behind Hilbert spaces. The first though goes back to the ancient Greek infinity:

[t]he simplest and most “natural” passage “from finiteness to infinity” is the “indefinite repetition” of the arithmetical operation of addition, on smaller and smaller summands, giving birth to the concept of convergent series, of which one can already find examples in Archimedes.” [125, p. 87–88].

An integral equation is an equation that has an unknown function under an integral sign. One of the simplest integral equations, leaving aside mathematical rigour for the purposes of clarity, is the equation

$$\int_a^x u(t)dt = f(x) \quad a \leq x \leq b,$$

where $f(x)$ is a given (differentiable) function, and $u(x)$ is the unknown function. From an introductory course in differential equations its solution is $u(x) = f'(x)$, where $f'(x)$ is the first derivative of f (with respect to x). The *Fredholm integral equation*, though is more difficult to solve:

$$u(x) + \int_a^b K(x,t)u(t)dt = f(x) \quad a \leq x \leq b,$$

where $K(x,t)$ is the *kernel function* of the integral equation. Both $K(x,t)$, and $f(x)$ are known, and $u(x)$ is the unknown function. David Hilbert showed in 1906 that solving the Fredholm integral equation was the same as solving the following *infinite* system of linear equations (see [52, p. 208]):

$$u_i + \sum_{j=1}^{\infty} K_{i,j}u_j = f_i \quad i = 1, 2, \dots,$$

where $\{u_i\}_{(i \in \mathbb{N})}$ is an unknown sequence of (real) numbers, $\{f_i\}_{(i \in \mathbb{N})}$ is a given sequence of (real) numbers, and $K_{i,j}$ ($i, j \in \mathbb{N}$) are the (real) entries of a given infinite matrix. Most importantly, though, was that the only

sequences-solutions to the Fredholm equation were *square-summable*, that is, of the form

$$\sum_{j=1}^{\infty} u_j^2 < \infty.$$

The set of all square-summable sequences of (real) numbers is denoted by $\ell^2(\mathbb{R})$.

By the time Riemann was making his contributions, the concept of “space” in mathematics had by far already been changed during the last fifty years before Riemann:

[w]ithout doubt, the spectacular development of various geometries during the 19th century, beginning with non-Euclidean geometries (Gauss, Lobachevsky, Bolyai) and culminating in 1872 in Klein’s Erlanger Programm, had profound influence on the idea of a general “space” [48, p. 262].

Non-Euclidean geometries, in fact, dispelled the concept of the parallels in a Euclidean geometry setting. The famous fifth postulate in Euclid’s *Elements*, in simple words, declared that if we are given a straight line on the plane, then from a given point that does not lie in the given line we can draw *only one* straight line parallel to the originally given line. On the other hand, Felix Klein’s main idea in his *Erlangen Programm* was that geometrical concepts should be studied using the theory of groups, symmetric groups in particular, rather than the traditional ruler and pair of compasses. Hilbert, influenced by the new intellectual climate on the concept of a geometrical space, proceeded with certain definitions that amount to the later concept of a dot product. In particular, if we consider the infinite (real) sequences $\vec{u} = \{u_i\}_{(i \in \mathbb{N})}$ and $\vec{v} = \{v_i\}_{(i \in \mathbb{N})}$ as vectors in the set of square-summable sequences $\ell^2(\mathbb{R})$, then by defining the *bilinear form*

$$\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{\infty} u_i v_i$$

we automatically have a *dot*, or rather *inner*, product. The square-summability condition then becomes

$$\langle \vec{u}, \vec{u} \rangle = \sum_{i=1}^{\infty} u_i^2 < \infty$$

which in fact is the Euclidean distance of \vec{u} with coordinates (u_1, u_2, \dots) if we consider the set of square-summable (real) sequences $\ell^2(\mathbb{R})$ as an infinite-dimensional Euclidean geometrical space. After Hilbert's publications major contributors in the field, such as Maurice Fréchet and Erhard Schmidt, started using the language of Euclidean geometry when referring to Hilbert spaces.

The interesting fact about the Soviet tradition in mathematics was that, in general, it seems to have continued Hilbert's approach to functional analysis, that is, functional analysis was presented to undergraduate students as a generalisation of Euclidean geometry. Integral equations, though, the main motivation behind Hilbert's contributions, comes from physics, and integral equations appeared as the *inverse* of differential equations. In American and Canadian (as well as in some British) mathematics departments, on the contrary, functional analysis was most often related to applications, especially in physics and engineering. In other words, Soviet mathematics focused more on the *pure* side of mathematics, while the majority of their North-American counterparts focused more on the *applied* side. As one Soviet mathematician remembers in a book on functional analysis and mechanics:

[i]n Russia, a university Mechanics department will typically exist within a “Mathematical Faculty.” Such a department is not an engineering department in the western sense, but is something intermediate between a mathematics department and an engineering department. [...] When the first author of this book [Leonid Lebedev] was a student [in Rostov State University] of the second author [Iosif Vorovich], functional analysis was not in the curriculum for mechanicians. In 1971, Professor Vorovich offered a short course on functional analysis to a broad audience consisting of mathematicians and mechanicians, students and professors.

It included a simple and minimal introduction to the theory of Banach and Hilbert spaces that opened the door to understanding (with some difficulty on the part of the non-mathematicians) certain interesting applications in mechanics. The mathematicians were *surprised* at how abstract theorems could be applied to mechanics and, moreover, that these theorems could actually be rooted in mechanics [275, p. v, italics added].

On the other hand, in the preface of a textbook still very popular in North-American, as well as British, functional analysis undergraduate courses, published in 1978, the author mentions:

Functional analysis plays an increasing role in the applied sciences as well as in mathematics itself. Consequently, it becomes more and more desirable to introduce the student to the field at an early stage of study. This book is intended to familiarize the reader with the basic concepts, principles and methods of functional analysis and its applications [261, p. v].

Very indicative of the Soviet school of mathematics was Andrei Kolmogorov's foundational work on probability theory. Probability theory already had a quite long history in mathematics, especially before Kolmogorov (see [408]). As it is well known among probabilists and statisticians

[p]robability theory has a right and a left hand. On the right is the rigorous foundational work using the tools of measure theory. The left hand "thinks probabilistically," reduces problems to gambling situations, coin-tossing, motions of a physical particle [65, p. ix].

It is indeed much easier to explain to a lay person, or to an undergraduate student of sociology, the meaning of the Central Limit Theorem, than to try to explain its proof, which is quite advanced, even for mathematicians. When Kolmogorov published in 1933 his *Grundbegriffe der Wahrscheinlichkeits Theorie* (see [252] for its English translation) the mathematics world was already

mature enough to embrace it. Probably the most striking aspect was its *axiomatic approach* to mathematical concepts of probability, similar to Euclid's elements. A practitioner's intuition was not enough, any more; a mathematically more *rigorous* approach was necessary. The axiomatic presentation of probability meant basically one thing: no matter how the practitioner defined probability relations, as long as they satisfied the axioms presented by Kolmogorov, they were mathematically sound. This rigorous tradition in mathematical probability seems to have been continued in the Soviet Union, as it can be seen in the English introduction to a translated, *undergraduate by Soviet standards*, text in stochastic processes, published in English in 1969, and required by Lomonosov as well as Kiev Universities, in the Soviet times as a prerequisite, as well as today in its second edition, to proceed to postgraduate study in probability theory and statistics in a Soviet university or research mathematics institute:

[t]he book is appropriate for students who have a sound background in probability from a measure-theoretic viewpoint and will, undoubtedly, be welcome as a graduate text. [...] The authors take great care to state the *topological assumptions* underlying each theorem, although occasionally a result may be stated in slightly greater generality than seems warranted by the proof. The book contains a wealth of results, ideas, and techniques, the deepest appreciation of which demands a most careful reading. Certainly, this is not a book for the indolent [171, p.iii, italics added].

Simply, or rather simplistically, put no probabilist or statistician studies, especially to a great depth, the topological properties of proofs in probability theory: that is usually done in the West by pure mathematicians; and probability measures in topological spaces, although formally a branch of probability, in the Western universities this is considered more as belonging to fields of real and complex analysis from a postgraduate viewpoint, that is, to the field of pure mathematics, rather than to the fields of probability theory and statistics, that is, to the field of applied mathematics.

One major change in nineteenth-century mathematics that had a major impact on modern pure mathematics was the rise of *structural* algebra:

the fact that algebraic manipulations on different kinds of “objects” had a strikingly similar appearance soon attracted attention, and after 1840 it gradually became clear that the essence of these manipulations did not lie in the *essence* of the objects, but in the *rules* to be followed in handling them, which might be the same for handling them [125, p. 116, italics in the original].

We saw earlier, for example, that the length of a vector in a Hilbert space is the infinite sum of the squares of its coordinates, just like the length of a vector in a three-dimensional Euclidean space is the sum of the squares of its three coordinates; the infinity of the sum of squares in a Hilbert-space vector, simply demands a more careful algebraic definition. According to Leo Corry

[t]wo domains of discourse can accordingly be tentatively identified when speaking about any scientific discipline; they can be described schematically as the “body of knowledge” and the “images of knowledge.” [...] he body of knowledge includes theories, ‘facts’, methods, open problems. The images of knowledge serve as guiding principles, or selectors. They pose and resolve questions which arise from the body of knowledge, questions which are in general not part of and cannot be settled within the body of knowledge itself. The images of knowledge determine attitudes concerning issues such as the following: Which of the open problems of the discipline most urgently demands attention? What is to be considered a relevant experiment, or a relevant argument? What procedures, individuals or institutions have authority to adjudicate disagreements within the discipline? What is to be taken as the legitimate methodology of the discipline? [...] What is the appropriate university curriculum for educating the next generation of scientists in a given discipline? Thus the images of knowledge cover both cognitive and normative views of scientists concerning their own discipline [107, p. 3–4].

What Corry called “images of knowledge”, in this thesis have been called the *social imaginary* of the discipline, since they are socially constructed by material means of voice, gestures, and blackboard or paper writing.

We saw in the third chapter the importance of gestures in everyday mathematical communication, whether this is teaching, lecturing, exchanging opinions and so on. The idea of the social imaginary has been borrowed from Castoriadis and

[i]t is the unceasing and essentially *undetermined* (social-historical and psychical) creation of figures/forms/images, on the basis of which alone there can ever be a question of “something.” What we call “reality” and “rationality” are its works [84, p. 3, italics in the original].

Castoriadis’s social imaginary has its origins in Sigmund Freud’s *unconscious*, but not as centre of psychical dynamics, but rather as a source of human creativity. The focus of this thesis was not just mathematicians, as practitioners, which seems to be a widespread view. The main focus of this thesis, on the contrary, was on *inventor mathematicians*, as pioneers and producers of knowledge, rather than receptors and reproducers of knowledge. In other words, the community of research mathematicians was considered as a community of artists, rather than as a community of practitioners. A more radical approach, in other words, had to be invented and implemented. The problem with speech, and in general language as speech as it is singularly perceived, is that pioneering mathematicians have been overly schooled in grammar and syntax, which, in its turn, has lead them to self-disciplinary regimes concerning speech. Gestures, on the other hand, are an understudied subject, and the grammar of gesturing is not as widespread as the grammar of speech. The intention, in other words, was that of “going native” in the following way: just as the San people, for example, in the Kalahari Desert had not been schooled at all in speech grammar, in a very similar way the mathematicians interviewed had not been schooled in gesture stud-

ies.³⁰ By focusing, therefore on gesturing while explaining, and presenting myself as a quite competent former mathematics postgraduate student, I could achieve a more “enhanced” authenticity from their part: they would use language, imagery, and (cinematic) gesturing performance without much internal speech discipline, a quite common trait among academicians. My focus, in other words, were slips of the language, and these would be more easily observed as slips on the gesturing aspect of language. I could thus achieve more *individual* authenticity in their expositions of mathematics. Another important component of this approach was asking the interviewees to elaborate on something that they themselves had contributed: that made them feel more comfortable, and another part of authenticity I wanted to induce is the authenticity someone gives off when walking in familiar territory, a territory that they themselves had created.

As one can infer from the previous sections of this chapter, the theory of Hilbert spaces is a rather demanding subfield of functional analysis, another very demanding field, as well. The intellectual efforts required of a sociologist of science without formal training in mathematics would potentially obstruct his or her observations. Let us not forget that what most probably protected Soviet mathematicians from the frequent interventions of the communist authorities into the Soviet mathematical community was, in fact, the high abstraction of mathematical knowledge: it demanded, and still demands, extensive training, its results are not visible to the general public, and amateur mathematicians with high claims of contribution practically do not exist. It can be safely said, in other words, that with the advent of the twentieth century self-taught professional mathematicians have practically disappeared, especially in the Soviet case. And this poses a problem in appreciating modern mathematics. Although one can “see” and “touch” technical artefacts in other disciplines, such as physics, chemistry, or biology, in mathematics, as the (generally quite simple for modern standards) example of Hilbert spaces has demonstrated, it is quite difficult to create a “museum” of mathematical artefacts such as a Hilbert or a Banach space, a simplex,

³⁰Only maybe they could have a passing acquaintance with “body language” lay perceptions from the popular press.

or an algebraic variety. The *material footprint* of mathematical artefacts, in other words, is very limited, when compared to other disciplines.

Chapter 6

Mathematics in Perspective

6.1 Theoretical Origins and Empirical Investigations

The theoretical inspirations behind this dissertation were Parsons's *The American University*, and Leroi-Gourhan's *Gesture and Speech*. I was interested, in other words, in an empirical framework for both of these theoretical underpinnings. Presenting Leroi-Gourhan's ideas was rather easy, since today there is an implicit consensus among the archaeological community that speech, gestures and stone tool construction and use are more deeply connected than previously thought. The only empirical challenge that remained was that with gesturing, though this was greatly enhanced by the recently emerged field of gesture studies. Empirical evidence, therefore, was to be based on video recordings, as sound recordings were considered as inadequate. Parsons, on the other hand, posed a challenge, since he never provided himself empirical evidence, and his main purpose was theoretical: to merge sociology and economics together. Although the final result was a little clumsy, his ideas were rather inspiring for the author, as well as challenging. Parsons's language is, in general, rather difficult to penetrate, as well as too stilted to ease reading and understanding it. So, instead of trying to apply Parsons's rather vague empirical framework, I let Parsons's idea of "the banking of sci-

ence” reside on the background of my mind waiting for an inspiration, and engaged my working attention to more empirical matters. But I will come to this in a while.

The most urgent problem to be solved was setting up a list of former Soviet mathematicians and contacting them. An initially helpful solution came from the *The Mathematics Genealogy Project*, an online database of academic genealogies of academicians (see [228]). By finding, therefore, one Soviet mathematician, I could also find all of his or her acknowledged academic descendants. Still though, there was another problem cropping up: there were no contact details, as well as no information as to where each mathematician was working, or whether he or she had passed away. Fortunately, later on, I discovered another database: www.mathnet.ru. [mathnet.ru](http://www.mathnet.ru) is an online database, set up originally by the Steklov Mathematics Institute, which contains, in Russian as well in English, all mathematicians who have published at least one scientific article in Russian, as well as mathematicians working in Russian and former Soviet Union universities: contact details, all the Russian articles authored by a Russian speaking mathematician, as well as hosting the online versions of all the Russian mathematical journals. The majority of the scientific articles on mathematics are free to download, except for those published the last one or two years, depending on the journal’s rules. This database is an extremely helpful one for sociologists and historians of mathematics. The problem of finding local archives was rather easily solved: there is the online database *ArcheoBiblioBase* set up by the International Institute of Social History in Amsterdam which can be found on www.iisg.nl/abb/index.php. *ArcheoBiblioBase* contains the electronic addresses of almost all the main central and regional archives of Russia. Since now most of the archives have online contact details any researcher can call the archivists up and ask them for more information. Still though, many online sites of the Russian archives are already very helpful.

A major problem that emerged during fieldwork was contacting potential interviewees in Russia. Before fieldwork I had attempted to contact many potential interviewees. Those in North American universities replied promptly. But from those located in Moscow very few replied. Even when I started

fieldwork in Moscow, it was very difficult to contact many potential interviewees by telephone or by email, whenever these appeared on the official site of the university they were employed. After some time, when I had made quite a few attempts I managed to contact two mathematicians, and through them I was introduced to many others. Another major hindrance which contributed to this problem of contacting potential interviewees was the fact that in the universities and the research institutes it is necessary to acquire a special invitation to enter the building, because there is a security guard. One cannot just enter the building and explore it. This experience actually prompted, later on, during the writing up of the thesis, the section on presenting the Soviet mathematics community as a social estate, rather than as a community. During fieldwork in Canada and the USA, on the contrary, everything went according to original planning. Almost all the emails I had sent were answered rather promptly, and most of the interviews were conducted according to my original scheduling. The interesting thing was that while in Canada and the USA I covered a much greater geographical area (practically all the East Coast) than in Russia (I was located in Moscow), in Moscow I had greater difficulty in accessing the community.

The research method I had chosen for the research was my version of grounded theory. My version means that I had already some preconceptions about what theoretical framework I wanted to follow, that is, a Parsonian one, it was, though, very vague in its details and I had to take a broad view on which types of data were relevant. I decided, in other words, to let Parsons's economic approach to stand in the background of my mind, and I started analysing the data of every interview after I had conducted it. From my first interviews I was handwriting notes on the interviews and contemplating on how to code them, and sometimes I revised them. I had already attended two seminars on NVivo at the university before I embarked on fieldwork, but at the time I did not want to use computer software, because I thought I would lose an intimate connection I had built up with my data. Later on, though, when I had to analyse all the 48 interviews I had conducted, in addition to the fieldnotes I had written, writing by hand became a burden that was delaying the writing up of the thesis. I decided, as a result, to start using

NVivo for data analysis to speed up the process of writing up. While in the beginning I experimented with a great number of codes and themes, in the end I fixed on a small number of them. One important was “gift”, whenever I saw enthusiasm in the fieldnotes when talking about mathematics, or a job in the Soviet Union irrelevant to the interviewee’s doctoral research in mathematics. Another code was “privilege” whenever I saw special objective privileges as a result of being a mathematician, or privileges perceived as such when, for example, choosing mathematics over other fields. A quite recurring phrase was “. . . a problem to solve . . .”, which in the end it became a code in itself as “lifestyle”.

The practicality of fieldwork research posed some limitations on the types of data collected. Initially I had in mind to conduct video recordings on each interview. This proved to be almost impossible due to various reasons. One was that it was my first time using a video camera and in many interviews I simply had not placed the camera in a proper position for capturing the details I wanted; or something unexpected would happen, such as the visit of the interviewee’s student or a colleague, and the interviewee would change position after the interruption and I had to readjust the camera. Another problem that emerged was that many interviewees were camera-shy, and did not want to be video-recorded; unfortunately I had not developed sufficiently compelling arguments as to why video-recordings were necessary for historical research in order to convince the interviewees to be video-recorded. Another data limitation was the use of the archives. After some engagement with archives, I decided to drop them as a source of data, because I realised that these types of data were not the ones I was looking for: I was interested in the social imaginary behind mathematics research, and that was something that only interviews could provide. Besides, I came to the conclusion that interviews could provide me with enough data to form a good picture of the culture behind mathematics research in the Soviet Union. The economic approach, though, that emerged later on during the writing up of the thesis pointed, in fact, to potential future projects where archival data could be more relevant: building up more quantitative models of mathematics research in the Soviet Union, as well as in other sociological fields, such as the political

social system, the legal social system, and so on, in a purely Parsonian spirit.

Some final comments on the dissertation as a whole seem to be in order. This particular project, or rather the final product of this research project, could seem either as quite intimidating or as rather ambitious to a doctoral student, depending on his or her self-confidence. I saw it as a quite ambitious one: it is indeed quite wide ranging. But my role models have been two towering figures of social research, that is, Jürgen Habermas and Michel Foucault: Habermas developed a theoretical framework that will last for centuries, while Foucault changed the face of sociology forever. I have never considered these two only as sociologists, but as professionals and practitioners in a profession with strong claims in improving and changing society. I decided to be ambitious because of the careers of these two particular sociologists. The most, probably, ambitious chapter in the thesis is chapter 2: it is a chapter that I went beyond my discipline, and indeed many claims against it could be made. But, in the end of the day, what scientific work in the social sciences is so perfect that no claims can be raised against its scientific validity?

The guarded reader should bear, also, in mind some important things in my opinion. When writing a thesis there are always back-end processes taking place, such as motivation and creativity. The reader, though, sees only the front-end product. As an aspiring scientist I had to set some standards for myself along with those that my future peers would set for me. I had been thinking for many years, before I started on my PhD, on the evolutionary perspectives of society, and many other ideas presented in this thesis were not formed during only the years of my PhD, but, actually, quite earlier. So this PhD was a chance for me, not only to acquire a doctoral degree, but also to organise my thought; *nobody else would have done for me, and for the next six to seven years I will most probably not have the time to do it again*. Now, on the contrary, I can more easily read articles on neurobiology and archaeology and relate them to my discipline. Maybe later, in my career to come, I will change my approach; maybe I will be able to make it more rigorous; maybe I will drop it altogether. Habermas did it twice; Foucault did it almost with every one of his books. A PhD is a journey, but in the

end of it its only visible part is the thesis. Everybody, though, seems to forget the most important goal, in my opinion, of a doctoral degree: not only to produce an original contribution to scientific knowledge, but to help the student develop his or her *research intuition*; to develop, in other words, his or her judgement on making decisions as to which directions involve fruitful lines of research. And this is how I saw my research project: to train myself to become a prolific future researcher, and every single chapter in this thesis served exactly this purpose, among others.

6.2 Mathematics as Applied Technology

We saw in the third chapter how gestures can be used when talking about a mathematical theorem, or any technical subject in mathematics. The interviews were conducted without informing in advance the interviewees about the observation of their gestures, which is an important part of this thesis, because it demonstrates the nature of human consciousness: spontaneous gestures demonstrate that consciousness is indeed *cinematic*. When in discussion about a mathematical object, the movements of gestures are employed as a secondary means in explaining, along with the sound of speech. Mathematicians' brains are no different from the brains of humans who lived twenty thousand years ago. So gestures and speech are fundamental material modalities in human understanding and human communicating. A starting point, in other words, in studying mathematicians and their theorems, and in fact in studying any scientific discipline, would be to start from the universals of human understanding as dictated and described by the archaeological and biological disciplines, rather than to resort to speculative philosophical discourse as a starting point. Philosophical speculation is actually very useful in producing ideas, but very counterproductive in producing facts, and this is how it has been treated in this thesis. Gestures, as well as speech, presented a very dynamic view of consciousness, a cinematic one, which is quite different from the view that Science and Technology Studies (STS) have rather implicitly adopted, following a general trend in the social sciences. Mathematical objects are first and foremost *temporal objects* whose

existence is rather phantasmatic than actually real. When a mathematical object is presented as a hand movement, then it can be concluded that human consciousness constructs stability out of instability. In fact cinematic consciousness is in every aspect of human perception: we do not only daydream while awake, but also properly dream while asleep.

Dreaming and daydreaming, though, are personal and very intimate activities, and as such they can be very elusive to the individual herself, as well as invisible to the co-present individuals. Gestures and speech, on the contrary, *externalise* cinematic consciousness in a cinematic way: the material modalities of speech sound and gesture movement, either unconsciously or consciously, or even partly-consciously, make visible the internal consciousness of the speaker, and then the listener's consciousness stabilises the cinematic nature of knowledge transfer during a conversation, a presentation, or classroom instruction. What actually stabilises mathematical knowledge is, in fact, *a writing system* as well as *(written) mathematical symbols*. When a mathematical proof has been written down, it becomes visible to everybody, but not necessarily understandable. A written mathematical proof has externalised materially a mathematician's elusive imagination in a quite stable mode. Published books and journal articles become now the *external memory* of mathematics as a discipline. Nobody remembers everything, and nobody knows what has been happening in every mathematical field and subfield; any time though they can photocopy an article, borrow a book, or today download a book collection, and have access to it in a more domesticated environment: in the silence of the university library, in the comfort of their personal office, or, in the relaxing environment of their homes. The cinematic nature of gesture and speech becomes a *photographic* one: knowledge, while in the beginning was audiovisual, acquires with writing a purely visual material modality, and material movement has been disposed of; knowledge has now become a series of *cinematic screenshots*. As a consequence one can now claim that mathematics is in fact applied technology, rather than the reverse.

If a sociologist attempts to examine herself these screenshots of mathematical knowledge, then it can become a quite challenging task, if not impossible.

What is a rational variety and what is a differentiable manifold? What is a prime ideal and what is almost-sure continuity? In the end of the day, what is proof in an infinite-dimensional Hilbert geometrical space, and how is it possible to prove anything in that space, since there are actually no infinite-dimensional artefacts? If there are indeed infinite-dimensional artefacts, as modern quantum and relativity theories assert, then these are simply non-perceivable by the human eye, and they can only be imagined. It was demonstrated, for example, in the third chapter, that during an interview the speaker was talking of an n -dimensional simplex and at the same time he was describing the simplex with the movement of his right hand. And the listener also could clearly see this, because he already knew what a simplex was. And there are also many articles and books on n -dimensional simplices which are in fact stabilised screenshots of simplices. And while mathematical description and mathematical notation can be difficult to penetrate, hand gestures, on the contrary, are universal, and they make the argument purely cinematic, that is, closer to the primary level of human consciousness, than to the secondary level of consciousness induced by artefacts. When only the members of a rather closed community are in position to decipher their own artefacts, those artefacts that their community has constructed, then we should be talking about a community of shamans, rather than a scientific community. The primary way a sociologist, who is not initiated to mathematics, has at his or her disposal to decipher the spirit-world of these shamans of science is hand gestures: patterned hand movement and hand posture provide a visible access to a mathematician's imaginary space. In this way, a sociologist can be initiated into the scientific cult of mathematics, by starting to see himself or herself what the mathematicians' spirit-world is starting to look like.

The fifth chapter examined the construction of the geometrical infinite-dimensional Hilbert spaces. The materiality of the book as well as the materiality of symbols was shown to be of primary importance. Then it was explained how two kinds of infinities were invented, and later extended, by mathematicians. Then these infinities were employed to create a new kind of infinite-dimensional geometry, the geometry of Hilbert spaces. Hilbert spaces are direct descendants of the ancient Greek Euclidean geometry. The problem

with Hilbert spaces: the ruler and the compasses cannot be used with Hilbert geometry, in the same fashion as in the original two- and three-dimensional ancient Greek Euclidean geometry. Did this prevent mathematicians from continuing? Not at all; their collective creativity was, and still remains, unprecedented. The purpose of this chapter was, in fact, more to cause puzzlement and make the (uninitiated) reader perplexed, rather than to clarify Hilbert geometry, or to explain some obscure points in these geometrical spaces. A puzzled and perplexed reader would most often ask: “Why am I unable to understand? Is my intelligence rather limited? Or are these mathematicians insane and have lost touch with reality?” All these questions, though, miss the actual point: understanding mathematics means joining the altered states of consciousness of the shamanistic community of mathematicians. In fact, every scientific community has their own spirit-world, or rather fictional universe, which is commonly referred to as their technical terminology. It is not only terminology, since when asked, any scientist answers both in speaking as well as in gesturing. A scientific field, any scientific field, is first and foremost a *literary and artistic genre*. There is no mathematics without writing technologies, as there is no scientific, or artistic, field without its own artefacts, whether these are called laboratory, microscope, Large Hadron Collider, paint, graphics, or calligraphy. Applied science does not seem to exist; what seem to exist are in fact *fields of applied technology*.

6.3 Scaling Up the Enterprise

Mathematics as a scientific-religious discipline was practised by a small minority of people in ancient Greece which was in fact more akin to a secret society. In the Soviet Union, on the contrary, the business of scientific mathematics had scaled up to gigantic proportions. Due to the limited financial compensation system in Soviet science, Soviet mathematics had developed its own gift economy. Mathematical knowledge was considered valuable, as well as the person who produced it. It was considered worthwhile, in other words, to spend one’s free time in research and publications. The community encouraged it, the Soviet state did not pay any attention to it, and every-

body was happy. The most important gift donation to the community was the scientific research article, which carried the name of its author: the whole community could claim possession of the article, but only its author could claim ownership. Everybody in the community was welcome to donate, and those who donated more, were those who received more respect and recognition. A published scientific article, in other words was an *heirloom* shared by everybody, and as such, was a connection link between the older generations and the younger ones. In this way a gift economy developed which was parallel to the usual financial economy. And each article had economic value, if we define economic value as the importance a certain community ascribes to this value. The article, as an economic artefact, that is, as a gift, makes the value of mathematics public: somebody has expended a lot human labour to produce it, and the fact of publication declares this product of human labour as acceptable to the scientific community. In this way, the material visibility of a gift donation and a gift receipt affirms the gift as a material instance of the imaginary scientific value, and reproduces the community by reproducing scientific value in the imaginary realm. A gift, in other words, is *self-referential*: it receives value *from* the community and at the same time returns *back to* the community this value.

By entering the university as a student in mathematics, the undergraduate started to become initiated into the gift economy of Soviet mathematics. Besides the compulsory courses for everyone, there were seminars every year organised by the research staff, and the subject of these seminars was related to the current research of the seminar organiser. These seminars were optional individually, but a student had to choose a number of them. Later on the students chose an advisor-supervisor who was usually a seminar organiser. If a student wanted to continue to postgraduate studies, then he had to publish at least two articles. Usually the advisor gave the student an unsolved problem to solve, and the student had to reciprocate by returning an attempted solution. The years of university study in the Soviet Union were five, and these problems were actually the student's master's dissertation. This exchange of an unsolved problem and an attempted solution was in fact the socialisation of the student into the gift economy of Soviet mathematics.

There were of course the gift exchanges of examinations, for example, but the exchange problem-solution between the advisor and the student was on a rather more personal level, than on an institutional one. The students, in this way created an emotional bond with the institution of science by creating an emotional bond with their advisors. There was, it is true, an instance of an interviewee who was very negative about his own advisor. But the general pattern was that one described. This kind of recruitment infused more loyalty to the mathematical scientific enterprise both on the student side as well as on the supervisor side. This student-supervisor relationship at the same time produced the institutional imaginary future of the enterprise: the student would start thinking about postgraduate study, discuss with the supervisor and the supervisor would help the student either himself, or with some of his connections. Needless to say that any institution without an imaginary future is bound to extinction.

The gift economy of Soviet mathematics, though, was not enough to scale up the research operations: institutions which could inspire trust were necessary, in order to make the system all-encompassing, and therefore more impersonal. The scientific article, beyond its author(s), has the issuing institution on it, that is, the scientific journal. The issuing institution accorded scientific validity to the article, transforming it into a coin: the “heads” were the banking institution, that is, the scientific journal, and the “tails” were the scientific article itself as one monetary unit. The banking institution was necessary to scale up: Soviet mathematicians were producing more and more theorems and these theorems needed a vehicle to validate them and make them publicly available. One peculiarity of the Soviet system of scientific journals was that there was no peer reviewing. The burden therefore of the validation lay on the journal editor, that is, the “fiduciary” of the bank, and probably on any other potential reader of the article. In the Western universities, on the contrary, validation lay, and still lies, within the peer-reviewing process, by unknown, that is, anonymous peer reviewers.

And here is a peculiarity probably of the Soviet and modern Russian society in general. While the Western university system was an economic system relying on *banks*, the Soviet university system was an economic system rely-

ing on *bankers*. And this is a major difference, in general, between modern Russia and the West: Western capitalism is a capitalism of banks, that is, institutions inspire trust; while Russian, as well as Soviet, capitalism is a capitalism of bankers, that is, the person behind the institution inspires trust. The fact that modern Russia has been quite often accused of systemic corruption most probably validates exactly this, since widespread corruption in a state is, in fact, a gift economy: a bribe is a (material/imaginary) connection between the identity of the briber and the identities of those bribed; and the more bribes and favours one has allocated, the more people become indebted to him or her, and the greater a status of a banking institution he or she acquires. In the USA the central banking institution of the USA is the Federal Reserve Bank and the central banker is Ben Bernanke; in today's Russia, on the contrary, it could indeed be said without much exaggeration, that while the central banking institution of Russia is the *Central Bank of the Russian Federation* the central banker in Russia *Vladimir Putin*. While in the West there is *anonymous capitalism*, in Russia there seems to exist *eponymous capitalism*, and Soviet mathematics was no exception to that.

Being a mathematician in the Soviet Union meant a certain way of life, a particular lifestyle. Students aspiring to become mathematicians learned very early that the culture of Soviet mathematics, was especially valued in the Soviet society. One of the interviewees, for example, mentioned that even television programmes promoted the lifestyle of a scientist. Mathematicians had their own separate buildings, their own special research activities, their own sense of distinct identity: through universities future researchers were not only socialised, but also integrated into the wider Soviet society. Most of their friends were, and still are, Soviet mathematicians. Moreover they were in general more close to their colleagues, to a point that one could even speak of a brotherhood of scientists, rather than a community. A fundamental activity both in working and in leisure time was solving problems and proving theorems: rather than a sign of a student's diligence, it was more of a *consumption pattern*. Solving problems, besides being a systemic imperative for an ambitious student, was at the same time a pleasant leisure activity. The community was very rarely harassed by the communist regime or any

other outsider to the community. The only major problem, that was rather endemic in all Soviet institutions, was the selection process of new undergraduates and postgraduates: antisemitism seemed to have remained widespread, but still many Jewish mathematicians continued research, mostly because of their emotional and social bonds to the mathematical community, rather than to the institution they worked for. As a community of its own and separate from other communities, mathematicians in the Soviet Union seemed to have been a social estate, rather than a community, or even a social class: their lifestyle was a shared one, most of their friends were from mathematics, and they enjoyed some extra privileges in a society of a meager distribution of means and resources.

6.4 Growing a Stronger Strong Programme

This thesis aspires to continue the Edinburgh's Strong Programme in the sociology of science, with some, though, necessary updates, in the author's opinion. The author, in particular, considers the Luhmannian sociocybernetic approach as a logical continuation and a further strengthening of the Strong Programme. According to Bloor the Strong Programme "would be *causal*, that is, concerned with the conditions which bring about belief or states of knowledge" [54, p. 7, my emphasis]. One should be very cautious with the so-called "causal explanations", especially in light of a modern quantum field theory backed by empirical data which claims that the universe is indeterministic, that is, mathematical probability equations describe, or rather prescribe, its evolution over time. Gravity on the one hand has never been observed: only its effects on materiality have been observed, because *Homo sapiens* individuals can observe *only* material modalities, that is, sound, voice, smell, weight, and so on: the human body evolved biologically to survive in a certain ecosystem by perceiving and utilising the ecosystem's material modalities; it did not evolve to produce scientific knowledge. Second, any causal explanation has to be backed by existing empirical data on the archaeological, biological and brain sciences, as this thesis has demonstrated. Data from these sciences provide the *sociological universals* of any

social activity, and not only of the sociology of sciences. Sociological explanations, in contradistinction, are mainly speculative and data collection is in fact *data production*, a fact that seems to be repeatedly forgotten. And it is data production for another reason: the sociologist's, and in general, the scientist's *creative ability* and *feeling of subjectivity*.

All of science and technology studies has repeatedly treated scientists so far as either robots, or mentally retarded in the best case scenario. Instead of studying their creative abilities as a social agent of the unknown, it has studied innovation by studying in fact the *products of innovation*, rather than innovation itself. Creativity lies in an unknown area beyond the sociologist's visibility, and the only visible means available so far is the gesture analysis method proposed in this thesis. If, for example, a social scientist wants to study creativity with respect to Lego bricks, she would observe how the toddler uses the material to make constructions with his hands. Each Lego brick is a word, and video-recording the construction can unveil the cinematic consciousness of the toddler. The gestural analysis proposed in the third chapter can, in fact, be generalised, and studying the successive screenshot frames can reveal a lot about creativity on the spot. In the same way one could study a mathematician handling equations and formulas, and observe the way these formulas levitate in the imaginary. The most proper research fields to study the innovator in connection with his or her innovation product are the psychoanalytic disciplines, art theory, and literary studies, and less the cognitive sciences. Gesture analysis for example, in this thesis, was conducted with an ethnographic-literary, rather than a cognitive-statistical, flavour.

The second tenet of the Strong Programme is that “[i]t would be *impartial* with respect to truth and falsity, rationality or irrationality, success or failure” [p. 7, my emphasis]. Since this thesis follows a Luhmannian social systems approach in its foundational principles, truth or falsity, or rationality or irrationality, and so on, are treated as *operational formalities* rather than as value systems that a social scientist should adhere to. What is the relation between truth in mathematics, truth in physics, or truth in chemistry: absolutely no relation. If a chemist proves the outcome of an experiment by means of a mathematical theorem and then tries to publish his or her

“existence proof” he or she will most probably will be scoffed at both by his colleagues, and most of his peers. If a physicist conducts an experiment to prove a mathematical theorem, and then tries to publish the results, he will be ridiculed, most probably, by the mathematical community. In a similar way, legality or illegality is again a binary operational formality outside science, that is, in the legal social system, with its own operational imperatives, depending on which subfield of the social legal system the social scientist is conducting research. But still, a creative scientist can create bridges between disciplines and thus pioneer into new cross-disciplinary and interdisciplinary fields such as modern cognitive archaeology, psycholinguistics, or evolutionary psychoanalysis: these three fields do produce scientific truth and facts, but these belong to the semantics of the corresponding subfield and its fictional technical universe, which its researchers adhere to. Each scientific field has its own accounting system as to which fact and which instance of truth belong to it: binary distinctions, in other words of that kind lead to operational differentiation the social system under scrutiny from its social environment: truth has to do with the social system of science, legality has to do with the social system of law, and so on. In the case of Soviet mathematics we saw that the social system of mathematics was in fact a social estate, and due to the lack of social mobility it was relatively easy for the system to differentiate itself from its social environment.

The third tenet of the Strong Programme declares that “[i]t would be *symmetrical* in its style of explanation” [p. 7, my emphasis]. The approach adopted in this thesis was that of a *sociology of a scientific cult*, rather than that of a sociology of scientific knowledge. The semantics of

cult identifies a pattern of ritual behavior in connection with specific objects, within a framework of spatial and temporal coordinates. *Ritual* behavior would include (but not necessarily be restricted to) prayer, sacrifice, votive offerings, competitions, processions, and construction of monuments. Some degree both of recurrence in place and repetition over time of ritual action is necessary for cult to be enacted, to be practiced [11, p. 398, emphasis in the original].

Modern *ritual behaviour* of members of the scientific community would include going to their offices everyday, teaching on a regular basis, meeting with their undergraduates on a supervisory basis, examination diets, construction of new buildings, and so on. By default, and by definition, therefore, this thesis' approach leads to symmetrical explanations in science, by treating each scientific field or subfield as a cult, rather than as a scientific field proper, if that has ever existed. This is an illusion, in fact, inculcated in modern universities. "If modern monetarist economics, for example, was a scientific field, then why this financial crisis of 2008 started, in the first place", would be a rather common-sense question to ask. Moreover Marxism-Leninism proclaimed itself as "scientific communism", rather than a theory.

The fact, borrowed as such from archaeology, is that science has existed for three centuries, while shamanism has existed for, at least, forty thousand years. The brain has not evolved since, so what makes modern science so special? The brains of modern mathematicians, monetary economists, and quantum physicists are functionally *exactly* the same as that of prehistoric shamans, and no proper social scientist can blatantly ignore that. Besides, by treating scientific disciplines as scientific cults leads to the idea of scientific knowledge as an *altered state of consciousness*. Only the shamans can "see" their spirit-world; only the mathematicians can "see" their Hilbert spaces; only the physicists can "see" their superstrings; only the economists can "see" their utility functions. And at this point, a social scientist can easily introduce the *social imaginary*, not as reflected on the artefacts, but as an imaginary proper, as a transcendental reality accessed only by the initiated, and as something that is elusive and at the same time organises society. Trust, for example, a necessity in any banking system, like that one described in the case of the Soviet science gift economy, lies on the imaginary realm, not on the real one. It cannot be measured by material instruments, but anyone can see the destructive lack of it: the stereotypical run on the bank.

Finally, the Strong Programme "would be *reflexive*" [p. 7, my emphasis]. Luhmannian sociocybernetics, in fact, satisfies this principle but with another name for it: *self-reference*. A rather older term for social reflexivity found in cybernetics would be that of a *feedback loop*. But modern sociology has

always been rather hypocritical with respect to cybernetics. All the sociology and anthropology undergraduates are being extensively instructed on *data* collection, *data* analysis, or big *data*; nobody, though, has ever acknowledged that the concept of “data” is an idea foreign to sociology; “data” has never been a sociological concept. On the contrary, in cybernetics, the precursor of modern computer science and informatics, it presupposes a sentient agent handling the appropriate data. In informatics departments there is another course vital for modern cyberneticians, that is, computer programmers: *data structures*, that is, how data are presented to a computing [or sentient] machine. There is, though, a certain institutional silence among the social sciences departments, as to the particular presentation and organisation of data, and how that affects the understanding of the sentient agents handling them, that is, the sociologists.

In the theoretical framework of this thesis, sociological data *are generated* and not collected, by *generating corresponding artefacts* as material data structures.¹ This is of fundamental importance, because under this light the social scientist is seen as the *agent* of explanation, and the data themselves are merely peripheral to the social research enterprise. So when Luhmann speaks of *second order observation* in the social sciences, what he means is that what is being observed, in fact, is not the social reality itself, but the material data structures produced during fieldwork. Due to the materiality of sociological data structures data have been accorded an illusion of objectivity, accompanied by an aura of an intimate connection with them. And when a sociologist produces a certain explanation for a social phenomenon, what he or she actually produces is not a causal, but a *narrative explanation* within a certain fictional universe which can very well be that of Latour’s sociological magic realism. Social scientists are no different from prehistoric shamans: they have produced a very elaborate and sophisticated spirit-world with its own fictional heroes, and its own distinct, and at times dividing, plots; it would be more advisable, though, that they were indeed more aware of it.

¹As shown in this thesis by data produced from one interview by sound as well as video recordings.

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